A UNIFIED LANGUAGE FOR
MATHEMATICS AND PHYSICS

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ABSTRACT. To cope with the explosion of information in mathematics and physics, we need a unified mathematical language to integrate ideas and results from diverse fields. Clifford Algebra provides the key to a unified Geometric Calculus for expressing, developing, integrating and applying the large body of geometrical ideas running through mathematics and physics.

INTRODUCTION

This first international workshop on Clifford Algebras testifies to an increasing awareness of the importance of Clifford Algebras in mathematics and physics. Nevertheless, Clifford Algebra is still regarded as a narrow mathematical specialty, and few mathematicians or physicists could tell you what it is. Of those who can, the mathematicians are likely to characterize Clifford Algebra as merely the algebra of a quadratic form, while the physicists are likely to regard it as a specialized matrix algebra. However, the very composition of this conference suggests that there is more to Clifford Algebra than that. Assembled here we have mathematicians, physicists and even engineers with a range of backgrounds much wider than one finds at a typical mathematics conference. What common insights bring us together and what common goal do we share? That is what I would like to talk about today.

The fact that Clifford Algebra keeps popping up in different places throughout mathematics and physics shows that it has a universal significance transcending narrow specialties. I submit that Clifford Algebra is as universal and basic as the real number system. Indeed, it is no more and no less than an extension of the real number system to incorporate the geometric concept of direction. Thus, a *Clifford Algebra is a system of directed numbers.* I have argued elsewhere [1] that Clifford Algebra is an inevitable outcome of the historical evolution of the number concept. Its inevitability is shown by the fact that it has been rediscovered by different people at many different times and places. You might prefer to say "reinvented" instead of "rediscovered," but with your leave, I will not attempt to distinguish between inventions and discoveries here.

Everyone here knows that the Pauli and Dirac algebras are Clifford Algebras invented by physicists to solve physical problems without knowing about Clifford's results. The same can be said about the invention of the Clifford Algebra of fermion creation and annihilation operators. I have personally witnessed several rediscoveries of Clifford Algebra which are not apparent in the literature. Two were enountered in papers I refereed. A more spectacular and fruitful rediscovery was made by the noted expert on combinatorial mathematics, Gian-Carlo Rota. He contacted me to find out how he could secure a copy of my book *Spacetime Algebra*; when I asked him why he was interested, he handed me a couple of his articles. These articles developed a mathematical system which Rota claimed to be ideal for the theory of determinants. I was astounded to discover that, except for trivial differences in notation, it was identical to the formulatlon of determinant theory

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which I had worked out with Clifford Algebra. In developing these articles, Rota and his coworker Stein had essentially rediscovered Clifford algebra,though Rota himself already knew Clifford Algebra very well. Of course, Rota must have suspected the connection and not taken the time to clear it up. The reinvention was not superfluous, however, because it produced a number of elegant results which I was able to incorporate in a broader treatment of Clifford Algebra [2].

I am sure that members of this audience can tell about other rediscoveries of Clifford Algebra. I think the discoveries of the complex numbers and quaternions should be included in this list, for I will argue later that they should be regarded as Clifford Algebras. Moreover, we shall see that even Clifford may not have been the original discoverer of Clifford Algebra.

The phenomenon of multiple discovery shows us that the development of Clifford Algebra is not the work of isolated individuals. It is a community affair. Excessive emphasis on priority of scientific discovery has contributed to the mistaken belief that scientific progress is primarily the work of a handful of great men. That belief does not take adequate account of the context of their discoveries, as indicated by the fact that the same scientific discovery is frequently made by two or more people working independently. Sociologist Robert Merton [4] has documented a large number of such multiple discoveries and argued convincingly that, contrary to conventional wisdom, multiple discoveries are far more common than discoveries by one person alone. Moreover, the more important the discovery, the more likely it is to be multiple. There is no better example than the multiple discoveries of Clifford Algebra.

The number is extremely large when all the significant variants of Clifford Algebra are included. This goes to show that genuine creative powers are not limited to a few men of genius. They are so widespread that discovery is virtually inevitable when the context is right. But the context is a product of the scientific community. So scientific discovery should not be regarded as an independent achievement by an individual. In the broad sense of discovery, Clifford Algebra should be seen as a common achievement of the scientific community.

I believe the phenomenon of multiple discovery is far more pervasive than historical accounts suggest. I hold with the Swiss psychologist Jean Piaget that the process of learning mathematics is itself a process of rediscovery. A similar view was expressed by the logician Ludwig Witgenstein, asserting in the preface to his famous *Tractatus Logico-Philosophicus* that he would probably not be understood except by those who had already had similar thoughts themselves. From that perspective, everyone at this conference is a codiscoverer of Clifford Algebra. That view may seem extreme, but let me point out that it is merely the opposite extreme from the view that mathematical discoveries are made by only a few men of genius. As usual, we find a deeper truth by interpolating between the extremes. Though we have many common discoveries, some of us had the benefit of broader hints. Mathematics progresses by making rediscovery easier for those who are learning it.

There may be no branch of mathematics in which more multiple discoveries can be found than in the applications of Clifford Algebra. That is because Clifford Algebra arises in so many "hot spots" of mathematics and physics. We will have many opportunities to note such discoveries, many of them by participants in this workshop. In fact, I don't believe I have ever attended a worhshop with so many multiple discoverers.

The subject of spinors, about which we will be hearing a lot, may be the hot spot which has generated the largest number of multiple discoveries. I have not met anyone who

was not dissatisfied with his first readings on the subject. The more creative individuals, such as the participants in this conference, are likely to try to straighten out matters for themselves. So off they go generating rediscoveries, most likely to be buried in the vast, redundant and muddled literature on spinors. One of the jobs of this conference should be to start straightening that literature out. Let us begin by noting that Professor Kahler [5] was the first among us to study the geometrical significance of representing spinors as elements of minimal ideals in the Dirac algebra.

Now I should explain my reason for bringing up the phenomenon of multiple discoveries. Though multiple discoveries may be celebrated as evidence for widespread creative powers of mankind, I want to emphasize that they are also symptoms of an increasingly serious problem, namely, the breakdown of communication among the different branches of science and mathematics. Needless duplication of effort, including multiple discoveries, is an obvious consequence of this breakdown. Moreover, progress will be seriously impeded when important results in one field are neither rediscovered nor transmitted to other fields where they are applicable.

During the last thirty years, the mathematics curriculum in universities of the United States has grown progressively less relevant to the training of physicists, so physics departments have taught more and more of the mathematics physicists need. This is only one of many signs of the growing gulf between physics and mathematics. But mathematics itself is becoming progressively more fragmented. Mathematicians in neighboring disciplines can hardly talk to one another. To mention only one example which I know very well, my father is an expert in the calculus of variations, but he has never communicated significantly with differential geometers, even in his own department, though their fields are intimately related. He is impatient with their jargon, for whenever he took the trouble to translate it, he found, they were saying something he already knew. So why bother? On the other hand, modern geometers tend to regard the calculus of variations as out of date, though they seldom reach its level of rigor and generality; even less do they realize that the field is still growing vigorously.

The language barriers between different branches of mathematics makes it difficult to develop a broad grasp of the subject. One frequently hears the complaint that mathematics is growing so rapidly that one cannot hope to keep up, let alone catch up. But how much of it is real growth, and how much is rediscovery or mere duplication of known results with a different nomenclature? New branches are being created, and names, definitions and notations are proliferating so profusely that the integrity of the subject is threatened. One gets the impression sometimes that definitions are being created more rapidly than theorems. Suffering from a Babylon of tongues, mathematics has become massively redundant.

As an example of redundancy even in traditional mathematics consider the following list of symbollc systems:

synthetic geometry	matrix algebra
coordinate geometry	Grassmann algebra
complex analysis	Clifford Algebra
vector analysis	differential forms
tensor analysis	spinor calculus

Each of these systems provides a representation of geometric concepts. Taken together,

therefore, they constitute a highly redundant system of multiple representations for geometric concepts. Let us consider the drawbacks of such multiple representation:

(a) **Limited access**. Most mathematicians and physicists are proficient in only a few of these ten symbolic systems. Therefore, they have limited access to results developed in a system with which they are unfamiliar, even though the results could be reformulated in some system which they already know. Moreover, there ls a real danger that the entire mathematical community will lose access to valuable mathematical knowledge which has not been translated into any of the current mathematical languages. This appears to be the case, for example, for many beautiful results in projective geometry.

(b) **Cost of translation**. The time and effort required to translate from one symbolic system to another is unproductive, yet translation is common and even necessary when one system is too limited for the problem at hand. For example, many problems formulated in terms of vector calculus or dlfferential forms can be simplified by re-expressing them in terms of complex variables if they have 2-dimensional symmetry.

(c) **Deficient integration**. The collection of ten symbolic systems is not an integrated mathematical structure. Each system has special advantages. But some problems call for the special features of two or more different systems, so they are especially unwieldy to handle. For example, vector analysis and matrix algebra are frequently applied awkwardly together in physics.

(d) **Hidden structure**. Relations among geometric concepts represented in different symbolic systems are difficult to recognize and exploit.

(e) **Reduced information content** (high redundancy), is a consequence of using several different symbolic systems to represent a coherent body of knowledge such as classical mechanics.

It goes without saying that elimination of these drawbacks would make mathematics easier to learn and apply. The question is, can it be done? And how can it be done without impairing the richness of mathematical structure? There is good reason to believe that it can be done, because all ten of the symbolic systems in our list have a common conceptual root in geometry. Historically, each system was developed to handle some special kinds of geometrical problem. To some extent, the creation of each system was a historical accident. But, from a broad historical perspective, each creation can be seen as an offshoot of a continuous evolution of geometric and algebraic concepts. It is time we integrated that accumulated experience into a unified mathematical system. This presents us with a fundamental problem in the design of mathematics: How can we design a coherent mathematical language for expressing and developing the full range of geometric concepts? In other words, how can we design a comprehensive and efficient *geometric calculus?* That is the problem I would like to discuss with you today. Before we get down to specifics, however, I would like set the problem in a general context.

The information explosion presents us with a staggering problem of integrating a wealth of information in science and mathematics so it can be efficiently stored, retrieved, learned and applied. To cope with this problem, we need to develop a *Metamathematical Systems Theory* concerned with the *design* of mathematical systems. The theory should explicate the principles and techniques of good mathematical design, including the choice of notations, definitions, axioms, methods of proof and computation, as well as the organization of theorems and results. I submit that one of the secrets of mathematical genius is being privy to powerful mathematical design principles which are not explicitly formulated in

the literature. By formulating the principles explicitly, they can be more readily taught, so people don't have to keep rediscovering them. Moreover, the principles can then be criticized and improved. We cannot afford the *laissez-faire* proliferation of mathematical definitions and notations. This is not to say that freedom in mathematical design should be in any way restricted. But we need rational criteria to distinguish good design from poor design. Custom is not enough.

In computer science the theory of software systems design is undergoing vigorous development. The design of mathematical systems is no more than software systems design at the highest level. So the theories of mathematical design and software design should merge continuously into one another. They should also merge with a new branch of applied mathematics called "Mathematical Systems Theory."

Returning to the specific problem of designing a Geometric Calculus, we note that this should take us a long way towards a general theory of mathematical design, for the corpus of geometric concepts underlies a major portion of mathematics, far exceeding the ten symbolic systems mentioned above. Actually, the design and development of Geometric Calculus is far along, far enough, I think, so it can be claimed without exaggeration to apply to a wider range of mathematics and physics than any other single mathematical system. I have been working on the development of Geometric Calculus for about 25 years, and a significant portion of the results are published in references [1], [2] and [3]. I am keenly aware that most of the results are translations and adaptations of ideas developed by others. I have endeavored to incorporate into the system every good idea I could find. But many ideas would not fit without some modification, so the process involved many decisions on the details of mathematical design. Although we cannot hope to cover many of the details, I would like to review with you today some of the most important ideas that have gone into the design of Geometric Calculus, and outline a program for further development of the language. As I see it, many participants at this workshop have already contributed to the development of Geometric Calculus, whether they have thought of their work that way or not. Anyway, the future development will require contributions from people with diverse backgrounds, such as we have here I hope many of you will join me to make this a community program.

1. DIRECTED NUMBER SYSTEMS

The first step in developing a geometric calculus is to encode the basic geometric concepts in symbolic form. Precisely what geometric concepts are basic is a matter of some dispute and partly a matter of choice. We follow a preferred choice which has emerged from generations of mathematical investigation. We take the concepts of magnitude and direction as basic, and introduce the concept of vector as the basic kind of directed number.

What does it mean to be a directed number? The answer requires an algebraic definition and a geometric interpretation. Directed numbers are defined implicitly by specifying rules for adding and multiplying vectors. Specifically, we assume that the vectors generate an associative algebra in which the square of every vector is a scalar. Multiplication is thus defined by the rules, for any vectors a, b, c,

$$a(bc) = (ab)c \tag{1.1}$$

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$$a(b + c) = ab + ac \tag{1.2a}$$

$$(b + c)a = ba + ca \tag{1.2b}$$

$$a^2 = \text{scalar} \tag{1.3}$$

Assuming that the scalars are real numbers, the sign of the scalar a^2 is called the *signature* of the vector a. So the signature of a vector may be positive, negative or zero.

These simple rules defining vectors are the basic rules of *Clifford Algebra*. They determine the mathematical properties of vectors completely and generate a surprisingly rich mathematical structure. But the feature making Clifford Algebra stand out among algebraic systems is the geometric interpretation which can be assigned to multiplication. Although the interpretation is not part of the formal algebraic system, it determines how the system is developed and applied, in particular, how it is applied to physics. The interpretation turns Clifford Algebra into the grammar for a language describing something beyond itself.

The geometric interpretation is most apparent when the product ab is decomposed into symmetric and antisymmetric parts by writing

$$ab = a \cdot b + a \wedge b, \tag{1.4}$$

where

$$a \cdot b = \tfrac{1}{2}(ab + ba), \tag{1.5}$$

$$a \wedge b = \tfrac{1}{2}(ab - ba). \tag{1.6}$$

We may regard (1.5) and (1.6) as definitions of subsidiary products, a symmetric *inner product* $a \cdot b$ and an antisymmetric outer product $a \wedge b$. The product $a \cdot b$ is just the usual scalar-valued inner product on a vector space, and its geometric interpretation is well known to everyone here. The quantity $a \wedge b$ is neither vector nor scalar; it is called a *bivector* (or 2-vector) and can be interpreted as a directed area, just as a vector can be interpreted as a directed length. The outer product $a \wedge b$ therefore describes a relation between vectors and bivectors, two different kinds of directed numbers. Similarly the inner product $a \cdot b$ describes a relation between vectors and scalars. According to (1.4), these are combined in a single *geometric product* ab, which is itself a kind of composite directed number, a quantity which completely describes the relative directions and magnitudes of the vectors a and b. More details on the interpretation of the geometric product and its history are given in [1].

The inner and outer products were originated by H. Grassmann as representations of geometric relations. I learned from Professor Bolinder only a few years ago that, late in his life, Grassmann [6] added the inner and outer products to form a new product exactly as expressed by (1.4). Thus, Grassmann discovered the key idea of Clifford Algebra independently of Clifford and evidently somewhat before him. This has been overlooked by historians of mathematics, who have dismissed Grassmann's later work as without interest. Generalizing (1.6), for the antisymmetrized product of k vectors $a_1, a_2, \ldots a_k$, we write

$$A_k = a_1 \wedge a_2 \wedge \ldots \wedge a_k \tag{1.7}$$

This is exactly Grassmann's k-fold outer product. Unless it vanishes it produces a directed number A_k called a *k-vector*, which can be interpreted as a directed k-dimensional volume.

I will refer to the integer k as the *grade* of the k-vector. I have adopted the term "grade" here mainly because it is monosyllabic, and there are some problems with alternative terms such as "dimension." The relative direction of a vector a and a k-vector A_k is completely characterized by the geometric product

$$a A_k = a \cdot A_k + a \wedge A_k \tag{1.8}$$

where, generalizing (1.5) and (1.6),

$$a \cdot A_k = \tfrac{1}{2}(a A_k + (-1)^{k+1} A_k a) \tag{1.9}$$

is a $(k-1)$-vector and

$$a \wedge A_k = \tfrac{1}{2}(a A_k - (-1)^{k+1} A_k a) \tag{1.10}$$

is a $(k+1)$-vector. Thus, the inner product is a grade lowering operation while the outer product is a grade raising operation. The inner product differs from a scalar product in that it is not necessarily scalar-valued. Rather, the inner product (1.9) is equivalent to the operation of contraction in tensor algebra, since k-vectors correspond to antisymmetric tensors of rank k.

This brings up an important issue in mathematical design. Clifford algebras are sometimes defined as certain ideals in tensor algebras. There is nothing logically wrong with this, but I submit that it is better mathematical design to reverse the process and introduce tensor as multilinear functions defined on Clifford algebras. Two reasons for this. First, the geometric product should be regarded as an essential part of the definition of vectors, since it is needed for the interpretation of vectors as directed numbers. Second, Clifford algebras are more versatile than tensors in applications, as I believe is born out in practice.

I should point out that if we regard the geometric product as essential to the concept of vector, as required by the most basic design principle of geometric calculus, then we must distinguish vector spaces on which the product is defined from linear spaces on which it is not. We must eschew the standard use of the term "vector" for elements of arbitrary linear spaces. If a precedent is needed for this, recall that when Hamilton introduced the term "vector" he definitely regarded multiplication as part of the vector concept.

The notation for Clifford Algebras is not yet standardized in the literature, so for the purposes of this conference, let me recommend one employed by Porteous [7] and Lounesto [8]. Let $\mathcal{R}^{p,q}$ denote a vector space of dimension $n = p + q$ over the reals consisting of a p-dimensional subspace of vectors with positive signature orthogonal to a q-dimensional subspace of vectors with negative signature. In this context, "orthogonality" of vectors means a vanishing inner product. Let $\mathcal{R}_{p,q}$ denote the Clifford Algebra generated by the vectors of $\mathcal{R}^{p,q}$ and let $\mathcal{R}_{p,q}^k$ be the $\binom{n}{k}$-dimensional subspace of k-vectors in $\mathcal{R}_{p,q}$. Elements of the 1-dimensional subspace $\mathcal{R}_{p,q}^n$ are called *pseudoscalars* of $\mathcal{R}^{p,q}$ or $\mathcal{R}_{p,q}$. An arbitrary element of $\mathcal{R}_{p,q}$ is called a *multivector* or Clifford number.

For the important case of a vector space with Euclidean (positive) signature, it is convenient to introduce the abbreviations $\mathcal{R}^n = \mathcal{R}^{n,0}$, $\mathcal{R}_n = \mathcal{R}_{n,0}$, and $\mathcal{R}_n^k = \mathcal{R}_{n,0}^k$. An n-dimensional vector space $\mathcal{C}^n$ over the complex numbers generates a Clifford algebra $\mathcal{C}_n$ with k-vector subspaces $\mathcal{C}_n^k$.

I am of half a mind to outlaw the Complx Clifford Algebras altogether, because the imaginary scalars do not have a natural geometric interpretation, and their algebraic features exist already in the real Clifford Algebras. However, there is already a considerable

literature on complex Clifford Algebras, and they do have some formal advantages. For example, a student of mine, Patrick Reany, has recently shown [9] that the solutions of any cubic equation over the complex numbers are contained in the solutions X of the simple cubic equations $X^3 = C$, where X and C are elements of the complex algebra $\mathcal{C}_1$. This opens up new possibilities for the theory of algebraic equations. Note that $\mathcal{C}_1$ is the largest Clifford Algebra for which all the elements commute. But note also that $\mathcal{C}_1$ can be regarded as a subalgebra of the real Euclidean algebra $\mathcal{R}_3$, with the pseudoscalar $\mathcal{R}_3^3$ playing the role of imaginary scalars.

Clifford Algebras are worth studying as abstract algebraic systems. But they become vastly richer when given geometrical and/or physical interpretations. When a geometric interpretation is attached to a Clifford Algebra, I prefer to call it a *Geometric Algebra*, which is the name originally suggested by Clifford himself. In this case, $\mathcal{R}_{p,q}$ is to be regarded as a generalized number system, a system of directed numbers for making geometrical representations and computations. To emphasize the computational features of the number system, I like to call $\mathcal{R}^{p,q}$ *arithmetic (p,q)-space*.

2. GEOMETRIC FUNCTION THEORY

Having identified Geometric Algebra as a suitable grammar representing basic geometrical relations, the next step in developing a Geometric Calculus is to use this grammar to construct a theory of functions for representing complex geometrical relations and structures. I like to call this construction *Geometric Function Theory*. Of course we want to integrate into the theory the rich body of results from real and complex analysis. This presents us with a definite design problem. I hope to convince you that this problem has a beautiful solution which points the direction to a powerful general theory. The key to the solution is an analysis of the geometric structure implicit in complex variable theory.

It is customary for mathematicians and physicists to regard complex numbers as scalars, mainly I suppose because it is an algebraic completion of the real number system. From the broader perspective of geometric algebra, however, this custom does not appear to be a good decision in mathematical design. Nor is it consistent with the geometrical interpretation of complex numbers which motivated the historical development of complex analysis and is crucial to many of it applications. From the beginning with multiple discovery of complex numbers by Wessel and Gauss, the unit imaginary **i** was interpreted as the generator of rotations in a 2-dimensional space. Moreover, development of the subject was stimulated by physical interpretations of complex functions as electric fields, magnetic fields or velocity fields of fluid flow in a physical plane.

Despite the great richness and power of complex function theory, it suffers from a defect in design which makes it difficult to see how the theory should be generalized beyond two dimensions. Fortunately, the defect is easily repaired. The defect is an ambiguity in geometrical representation. Throughout the subject, complex numbers play two distinctly different geometrical roles which are not differentiated by the mathematical formalism. On the one hand, a complex number may represent a point in a plane; on the other hand it may represent a rotation-dilation. Moreover, the representation of points has the defect of singling out a preferred direction (the real axis) with no geometrical or physical significance in applications.

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The defect can be removed by regarding the "plane" of complex numbers as the even subalgebra $\mathcal{R}_2^+$ of the geometric algebra $\mathcal{R}_2$ for the "real plane" $\mathcal{R}^2$. Then we can write $\mathcal{R}_2 = \mathcal{R}^2 + \mathcal{R}_2^+$, exhibiting the entire algebra $\mathcal{R}_2$ as the sum of the real and complex planes. The crucial points of this construction is that the unit imaginary $\mathbf{i}$ is now to be interpreted as a bivector, the unit pseudoscalar for the real plane $\mathcal{R}^2$, but it is also the generator of rotations in the real plane. For it is easily proved any vector in $\mathcal{R}^2$ by 90°. Parenthetically, let me note that we could have achieved the same result by using the geometric algebra $\mathcal{R}_{0,2}$ instead of $\mathcal{R}_2 = \mathcal{R}_{2,0}$. But then the vectors would have negative square, which might be a drawback or an advantage, depending on what you want to do with it. The advantage is that $\mathcal{R}_{0,2}$ can be identified with the quaternions and the even subalgebra of $\mathcal{R}_3$.

A linear mapping of the real plane $\{\mathbf{x}\} = \mathcal{R}^2$ onto the complex plane $\{z\} = \mathcal{R}_2^+$ is defined by the equation $z = \mathbf{x}\mathbf{e}_1$, where $\mathbf{e}_1$ is a fixed unit vector specifying the direction which is mapped into the real axis. By the way, the same kind of linear mapping of vectors into the even subalgebra is of great importance in all Clifford Algebra, because it defines a linear relation between Clifford Algebras of different dimension. I call it a *projective mapping*, because it provides an algebriac formulation for the basic idea of introducing homogeneous coordinates in projective geometry. We shall encounter it again and again in this conference.

We can use this projective mapping between the real and complex plane to map functions of a complex variable to functions on the real plane where there is no preferred real axis. As an important example, we note that for an analytic function $F(z)$, Cauchy's Theorem

$$\oint dz\, f(z) = 0 \qquad \text{maps to} \qquad \oint d\mathbf{x}\, f(\mathbf{x}) = 0\,, \tag{2.1}$$

where $f(\mathbf{x}) = \mathbf{e}_1 F(\mathbf{x}\mathbf{e}_1)$ is a vector field on the real plane. Note that $d\mathbf{x} = dz\mathbf{e}_1$ is a vector-valued measure for the curve over which the integral is taken. I believe that this is the key to the great power of complex analysis, and we can transfer this power to all of real analysis if we use directed measures in defining multiple integrals. Let me explain how.

As you know, Cauchy's Theorem can be derived from Green's Theorem which, using geometric algebra can be written in the form

$$\int_{\mathcal{A}} dA\, \nabla f = \oint_{\partial\mathcal{A}} d\mathbf{x}\, f\,. \tag{2.2}$$

Here dA is an element of directed area, that is, a bivector-valued measure, so it can be decomposed into $dA = \mathbf{i}\,|dA|$ where $\mathbf{i}$ is the unit bivector. Also, $\nabla f = \nabla_{\mathbf{x}} f(\mathbf{x})$ is the derivative of f with respect to the vector $\mathbf{x}$. The operator $\nabla = \nabla_{\mathbf{x}}$ is algebraically a vector, because $\mathbf{x}$ is a vector. Therefore, we can use (1.6) to write $\nabla f = \nabla \cdot f + \nabla \wedge f$, which is a decomposition of the vector derivative into divergence and curl. Note that $dA \wedge \nabla = 0$, so $dA\nabla = dA \cdot \nabla$ has the same grade as $d\mathbf{x}$, which, of course, is essential if (2.2) is to make sense.

It is fair to ask how the derivative with respect to a vector should be defined. This is a design problem of the utmost importance, because $\nabla = \nabla_x$ is the fundamental differential operator of Geometric Calculus. The solution is incredibly simple and can be found by applying one of the most basic principles of mathematical design: Design your definitions to put the most important theorems in the simplest possible form. Accordingly, we define the differential operator ∇ so that (2.2) is true. We simply define ∇f at $\mathbf{x}$ by taking the

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appropriate limit of (2.2) as the set $\mathcal{A}$ shrinks to the point $\mathbf{x}$. This took me some time to realize, as I originally defined ∇ differently and then derived (2.2) with $dA \cdot \nabla$ on the left without realizing how to get rid of the dot (Ref. [3]). Note that this definition of the derivative is completely coordinate-free.

Equation (2.2) is the most important theorem in integral calculus. It applies not only to the real plane but to any two dimensional surface $\mathcal{A}$ in $\mathcal{R}^{p,q}$ with boundary $\partial\mathcal{A}$, with possible exceptions only on null surfaces. Indeed, it applies without change in form when $\mathcal{A}$ is a k-dimensional surface, if we interpret dA as a k-vector-valued measure on $\mathcal{A}$ and $d\mathbf{x}$ as a $(k-1)$-vector-valued measure on $\partial\mathcal{A}$. It thus includes the theorems of Gauss and Stokes, and it generalizes the so-called generalized Stokes' theorem from the theory of differential forms. To relate (2.2) to differential forms when $\mathcal{A}$ is a 2-dimensional surface, we simply take its scalar part. From the right side of (2.2) we get a 1-form $\omega = d\mathbf{x} \cdot f$ and from the left side we get the exterior derivative $d\omega = dA \cdot (\nabla \wedge f)$. So we get the generalized Stokes' theorem

$$\int_{\mathcal{A}} d\omega = \oint_{\partial\mathcal{A}} \omega \,. \tag{2.3}$$

The trouble with this is it is only half of (2.2). The other half is the bivector equation

$$\int_{\mathcal{A}} dA \, \nabla \cdot f = \oint_{\partial\mathcal{A}} d\mathbf{x} \wedge f \,. \tag{2.4}$$

It is true that this can also be expressed in terms of differential forms by introducing "dual forms," but the damage has already been done. By splitting (2.2) into parts we have diminished the power of the directed measure characteristic of complex analysis. With differential forms one cannot even write Cauchy's theorem (2.1) as a single formula. Differential forms bury the directed measure which is explicit in (2.2).

Equation (2.2) is so important that it should not be named after Stokes or any other person. It is the result of contributions by many people. In the 1-dimensional case when $\mathcal{A}$ is a curve with endpoints a and b, (2.2) reduces to

$$\int_a^b d\mathbf{x} \, \nabla f = \int_a^b d\mathbf{x} \cdot \nabla f = \int_a^b df = f(b) - f(a) \,. \tag{2.5}$$

This is widely known as *the fundamental theorem of integral calculus*. That is a suitable title for its generalization (2.2). There is more about this theorem in Ref. [2].

Returning to our reformulation of complex analysis on the real plane, we see that Cauchy's theorem (2.1) follows from the fundamental theorem (2.2) if and only if

$$\nabla f = 0 \,. \tag{2.6}$$

This implies that the divergence $\nabla \cdot f$ and curl $\nabla \wedge f$ vanish separately. It is, of course, just a coordinate-free form of the Cauchy-Riemann equations for an analytic functlon. Indeed, (2.6) is well-defined for any k-dimensional surface $\mathcal{A}$ in $\mathcal{R}^{p,q}$ on which ∇ is defined, and (2.2) then gives a corresponding generalization of Cauchy's Theorem. T herefore, (2.6) is a suitable generalization of the definition of an analytic function to k-dimensional surfaces. Moreover, the main theorems of analytic function theory generalize along with it. If the variable x is a point in $\mathcal{R}^n$, you might like to write $\nabla_x = \nabla$, since ∇^2 is the n-dimensional

Laplacian. Moreover, $\nabla f = 0$ implies $\nabla^2 f = 0$, from which one can conclude that the Cartesian components of an analytic function are harmonic functions.

If f is analytic except at a pole $\mathbf{x}'$ with residue q in $\mathcal{A}$, then $\nabla f = 2\pi q \delta(\mathbf{x} - \mathbf{x}')$, and substitution into (2.2) gives the "residue theorem"

$$\oint d\mathbf{x}\, f = 2\pi q \int_{\mathcal{A}} \mathbf{i} \,|\, dA(\mathbf{x}) \,|\, \delta(\mathbf{x} - \mathbf{x}') = 2\pi \mathbf{i} q \,. \tag{2.7}$$

I mention this to point out that the mysterious $\mathbf{i}$ in the residue theorem comes from the directed area element. This shows us immediately the generalization of the residue theorem to higher dimensions will replace the $\mathbf{i}$ by a unit k-vector.

Now I want to show that our reformulation of complex variable theory in terms of geometric algebra gives us someting more than the conventional theory even in two dimensions. One more reason for regarding $\nabla = \nabla_x$ as *the* fundamental differential operator is the fact that it has an inverse (or *antiderivative*) ∇^{-1}. Thus, if $f(x)$ is an unknown vector (or multivector) field whose derivative $j = \nabla f$ is unknown, then f can be obtained from $f = \nabla^{-1} j$ even when $j = 0$. Of course ∇^{-1} is an integral operator and boundary values of f must be specified to get a unique inverse. An explicit expression for ∇^{-1} can be derived from a modified form of the fundamental theorem (2.2). I give only the result which is derived in Ref. [2]:

$$f(x') = \frac{1}{2\pi} \int_{\mathcal{A}} |\, dA \,| \frac{1}{x - x'} \nabla f(x) + \frac{1}{2\pi i} \oint \frac{1}{x' - x}\, dx\, f(x) \,. \tag{2.8}$$

For an analytic function $\nabla f = 0$ in $\mathcal{A}$, this reduces to Cauchy's integral formula, where $\mathbf{e}_1 (x' - x)^{-1} = (z' - z)^{-1}$ will be recognized as the "Cauchy kernel." Thus, (2.8) is a generalization of Cauchy's integral formula to functions which are not analytic. Conventional complex variable theory does not find this generalization, because it is self-limited to the use of line integrals without area integrals. But the generalization (2.8) is valuable even in analytic function theory, because the area integral picks up the contributions from poles and cuts.

The generalization of (2.8) to higher dimensions is straightforward. One simply replaces the Cauchy kernel with an appropriate Green's function and performs directed integration over the specified surfaces. I will leave this topic to Professor Delanghe, who is one of the multiple discoverers of generalized Cauchy integral formula.

With the basic principles and theorems for vector differentiation and geometric integration in hand, the main business of Geometric Function Theory is ready to begin, namely, to develop a rich theory of special functions generalizing and integrating the magnificent results of classical function theory. This is an immense undertaking which will require an army of workers. Many participants at the conference have already contributed to the subject. One of the most elegant and valuable contributions is the generalization of the Moebius transformation by Professor Ahlfors, which we will hear about tomorrow. This fits into the program of Abel and Jacobi to develop a complete theory of algebraic functions and their integrals. However, geometric algebra gives us a much broader concept of algebraic function.

Let me suggest that "Geometric Function Theory" is a better name for this field than "Hypercomplex Function Theory" or other alternatives, because it lays claim to territory in the center of mathematics. A more esoteric name suggests that the field belongs on the periphery. I hope I have convinced you that in order to integrate complex analysis smoothly

into a general geometric function theory, one should regard complex numbers rather than scalars. In this general theory the artificial distinction between real and complex analysis disappears completely, and hypercomplex analysis can be seen as multidimensional real analysis with an appropriate geometrical tool.

3. SPACETIME CALCULUS

One of the main design principles for Geometric Calculus is that the language should be efficient in all practical applications to physics. For this reason, I have endeavored to incorporate into the language definitions, notations and conventions which are as close to standard usage in physics as is consistent with good mathematical design. That accounts for some of my deviation from perfectly good notations that mathematicians prefer. The calculus is quite well developed for all branches of physics. As an example with important implications, let me outline its application to classical relativistic electrodynamics. I aim to show three things. First, the calculus provides a compact coordinate-free formulation for the basic equations of the subject, namely, the electromagnetic field equations and energy-momentum tensor as well as the equation of motion for charged particles. Second, the calculus is an efficient computational tool for deriving consequences of those equations. Third, the classical formulation articulates smoothly with the formulation of relativistic quantum mechanics.

To begin, we formulate a model of physical spacetime in terms of geometric algebra. When the vectors of $\mathcal{R}^{1,3}$ are employed to designate points of flat spacetime, I call the geometric algebra $\mathcal{R}_{1,3}$ which it generates the *spacetime algebra* (STA), because I interpret it as a complete encoding of the basic geometrical properties of spacetime. It provides us with an arithmetic of spacetime directions and magnitudes. I leave it to Garret Sobczyk to discuss the generalization of this approach to curved spacetime.

For the derivative with respect to a spacetime point x, I like to use the notation $\square = \nabla_x$ to emphasize its special importance, and because its square is the usual d'Alembertian, for which $\square^2$ is a common notation. Of course $\square$ is a vector differential operator which can be defined in the same coordinate-free fashion we have already discussed.

An electromagnetic field is represented by a bivector-valued function $F = F(x)$ on spacetime. The field produced by a source with spacetime current density $J = J(x)$ is determined by Maxwell's field equation

$$\square\, F = J\,. \tag{3.1}$$

The field is found in essentially the same way we got Cauchy's integral formula, so let us write

$$F = \square^{-1} J\,. \tag{3.2}$$

The Green's functions and physical boundary conditions needed to determine $\square^{-1}$ have been thoroughly studied by physicists, mostly in a somewhat different language.

Comparison of our formulation (3.1) of Maxwell's equation with conventional formulations in terms of differential forms or tensors is easy, because the latter formalisms can be incorporated in Geometric Calculus simply by introducing a few notations. To do that, we define an electromagnetic 2-form ω by

$$\omega = \omega(\mathrm{d}x \wedge \mathrm{d}y) = (\mathrm{d}x \wedge \mathrm{d}y) \cdot F\,. \tag{3.3}$$

This is the projection of the electromagnetic field F onto an arbitrary directed area element $\mathrm{d}x \wedge \mathrm{d}y$. To get the tensor components of F, we choose an orthonormal basis $\{\gamma^\mu\}$ for $\mathcal{R}^{1,3}$. I use the notation γ^μ which is commonly employed for the Dirac matrices to emphasize the fact that they are different mathematical representations of exactly the same physical information I will elaborate on that point in my next lecture. Now, for the tensor components we get

$$F^{\mu\nu} = (\gamma^\nu \wedge \gamma^\mu) \cdot F = \omega(\gamma^\nu \wedge \gamma^\mu). \tag{3.4}$$

To relate the different formulations of the field equations, we use the decomposition $\Box F = \Box \cdot F + \Box \wedge F$ of the derivative into divergence and curl. The tensor form is obtained trivially by computing components with respect to a basis, using $\Box = \gamma^\mu \nabla_\mu$, for example. So we can concentrate on the formulation with differential forms. The curl is related to the exterior derivative d of a differential form by

$$\mathrm{d}\omega = \mathrm{d}V \cdot (\Box \wedge F), \tag{3.5}$$

where $\mathrm{d}V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is an arbitrary directed volume element. To relate the divergence to the exterior derivative we first relate it to the curl. This can be done by introducing the duality operation which, in the spacetime algebra, is simply multiplication by the unit pseudoscalar $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. We exploit the associative and distributive laws as follows:

$$\Box(Fi) = (\Box F)i,$$
$$\Box \cdot (Fi) + \Box \wedge (Fi) = (\Box \cdot F)i + (\Box \wedge F)i.$$

Separately equating vector and trivector parts, we obtain

$$\Box \cdot (Fi) = (\Box \wedge F)i, \tag{3.6a}$$
$$\Box \wedge (Fi) = (\Box \cdot F)i. \tag{3.6b}$$

We need only the latter equation. Defining a dual form for ω by $^*\omega = (\mathrm{d}x \wedge \mathrm{d}y) \cdot (Fi)$, we find the exterior derivative

$$\mathrm{d}^*\omega = \mathrm{d}V \cdot (\Box \wedge (Fi)) = \mathrm{d}V \cdot ((\Box \cdot F)i). \tag{3.7}$$

From the trivector Ji we get a current 3-form $^*\alpha = \mathrm{d}V \cdot (Ji)$, which is the dual of a one-form $\alpha = J \cdot \mathrm{d}x$. Now we can set down the three different formulations of the field equations for comparison:

$$\Box \cdot F = J \qquad\qquad \nabla_\mu F^{\mu\nu} = J^\nu \qquad\qquad \mathrm{d}^*\omega = {}^*\alpha$$
$$\Box \wedge F = 0 \qquad\qquad \nabla_{[\alpha} F_{\mu\nu]} = 0 \qquad\qquad \mathrm{d}\omega = 0$$

I leave it for you to decide which form is the best. Of course, only the STA form on the left enables us to unite the separate equations for divergence and curl in a single equation. This unification is nontrivial, for only the unified equation (3.1) can be inverted directly to determine the field as expressed by (3.2).

It should be obvious also that (3.1) is invariant under coordinate transformations, rather than covariant like the tensor form. The field bivector F is the same for all observers; there

is no question about how it transforms under a change of reference system. However, it is easily related to a description of electric and magnetic fields in a given inertial system in the following way:

An inertial system is determined by a single unit timelike vector γ_0, which can be regarded as tangent to the worldline of an observer at rest in the system. This vector determines a split of spacetime into space and time, which is most simply expressed by the equation

$$x\gamma_0 = t + \mathbf{x}\,, \tag{3.8}$$

where $t = x \cdot \gamma_0$ and $\mathbf{x} = x \wedge \gamma_0$. This is just another example of a projective transformation like the one encountered in our discussion of complex analysis. It is a linear mapping of each spacetime point x into a scalar t designating a time and a vector $\mathbf{x}$ designating a position. The position vectors for all spacetime points compose a 3-dimensional Euclidean vector space $\mathcal{R}^3$. I denote the vectors in $\mathcal{R}^3$ with boldface type to distinguish them from vectors in $\mathcal{R}^{1,3}$. The equation $\mathbf{x} = x \wedge \gamma_0$ tells us that a vector in $\mathcal{R}^3$ is actually a bivector in $\mathcal{R}_{1,3}^2$. In fact, $\mathcal{R}^3$ consists of the set of all bivectors in $\mathcal{R}_{1,3}^2$ which have the vector γ_0 as a factor. Algebraically, this can be characterized as the set of all bivectors in $\mathcal{R}_{1,3}^2$ which anticommute with γ_0. This determines a unique mapping of the electromagetic bivector F into the geometric algebra $\mathcal{R}_3$ of the given inertial system. The space-time split of F by γ_0 is obtained by decomposing F into a part

$$\mathbf{E} = \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \tag{3.9a}$$

which anticommutes with γ_0 and a part

$$i\mathbf{B} = \tfrac{1}{2}(F + \gamma_0 F \gamma_0) \tag{3.9b}$$

which commutes with γ_0, so we have

$$F = \mathbf{E} + i\mathbf{B}\,, \tag{3.10}$$

where, as before, $i = \gamma_0\gamma_1\gamma_2\gamma_3$ is the unit pseudoscalar. Although $i\mathbf{B}$ commutes with γ_0, $\mathbf{B}$ must anticommute since i does. Therefore, we are right to denote $\mathbf{B}$ as a vector in $\mathcal{R}^3$. Of course, $\mathbf{E}$ and $\mathbf{B}$ in (3.10) are just the electric and magnetic fields in the γ_0-system, and the split of F into electric and magnetic fields will be different for different inertial systems. The geometric algebra generated by $\mathcal{R}^3$ is identical with the even subalgebra of $\mathcal{R}_{1,3}$, so we write

$$\mathcal{R}_3 = \mathcal{R}_{1,3}^+\,. \tag{3.11}$$

Moreover, (3.10) determines a split of the bivectors in $\mathcal{R}_{1,3}$ into vectors and bivectors of $\mathcal{R}_3$, as expressed by writing

$$\mathcal{R}_{1,3}^2 = \mathcal{R}_3^1 + \mathcal{R}_3^2\,. \tag{3.12}$$

This split is not unique, however, as it depends on the choice of the vector γ_0.

A complete discussion of space-time splits is given in Refs. [4] and [10]. My object here was only to explain the underlying idea. Physicists often write $E_k = F_{k0}$ to identify the electric field components of $F_{\mu\nu}$ without recognizing that this entails a mapping of spacetime bivectors into space vectors. We have merely made the mapping explicit and extended it

systematically to space-time splits of all physical quantities. Finally, I should point out that the purpose of a spacetime split is merely to relate invariant physical quantities to the variables employed by a particular observer. It is by no means necessary for solving and analyzing the basic equations. As a rule, it only complicates the equations needlessly. Therefore, the best time for a split is usually after the equations have been solved.

Now we turn to the second major component of electrodynamics, the energymomentum tensor. Geometric algebra enables us to write the energymomentum tensor $T(n)$ for the electromagnetic field F in the compact form

$$T(n) = \tfrac{1}{2} F n \widetilde{F} \,, \tag{3.13}$$

where the tilde indicates "reversion" and $\widetilde{F} = -F$. For the benefit of mathematicians in the audience, let me say that the $T(n)$ is a vector-valued linear function on the tangent space at each space time point x describing the flow of energymomentum through a hypersurface with normal n. The coordinate-free form (3.13) of the energymomentum tensor is related to the conventional tensor form by

$$\begin{aligned}
T^{\mu\nu} = \gamma^\mu \cdot T(\gamma^\nu) &= \tfrac{1}{2} \langle \gamma^\mu F \gamma^\nu \widetilde{F} \rangle &= -\langle \gamma^\mu \cdot F \gamma^\nu F \rangle - \tfrac{1}{2} \langle \gamma^\mu \gamma^\nu F^2 \rangle \\
&= -(\gamma^\mu \cdot F) \cdot (\gamma^\nu \cdot F) - \tfrac{1}{2} \gamma^\mu \cdot \gamma^\nu \langle F^2 \rangle &= F^{\mu\alpha} F^\nu_\alpha + \tfrac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \,,
\end{aligned} \tag{3.14}$$

where the angular brackets denote "scalar part."

The algebraic form (3.13) for $T(n)$ is not only more compact, it is easier to apply and interpret than the tensor form on the right side of (3.14). To demonstrate that, let us solve the eigenvector problem for $T(n)$ when $F^2 \neq 0$. In that case, it is easy to show (Ref. [3]) that F can be put in the canonical form

$$F = f e^{i\phi} \tag{3.15}$$

where f is a timelike bivector with

$$f^2 = [(F \cdot F)^2 + |F \wedge F|^2]^{\frac{1}{2}} \,, \tag{3.16}$$

and ϕ is a scalar determined by

$$e^{i\phi} = \left[\frac{F^2}{f^2} \right]^{\frac{1}{2}} = \frac{(F \cdot F + F \wedge F)^{\frac{1}{2}}}{[(F \cdot F)^2 + |F \wedge F|^2]^{\frac{1}{4}}} \,, \tag{3.17}$$

Since the pseudoscalar i commutes with all bivectors but anticommutes with all vectors, substitution of (3.15) into (3.13) reduces $T(n)$ to the simpler form

$$T(n) = -\tfrac{1}{2} f n f \,. \tag{3.18}$$

This is simpler because f is simpler, in general, than F. It enables us to find the eigenvalues by inspection. The bivector f determines a timelike plane. Any vector n in the plane satisfies the equation $n \wedge f = 0$, which implies that $nf = -fn$. On the other hand, if n is orthogonal to the plane, then $n \cdot f = 0$ and $nf = fn$. For these two cases, (3.18) gives us

$$T(n) = \pm \tfrac{1}{2} f^2 n \,. \tag{3.19}$$

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Thus $T(n)$ has a pair of doubly degenerate eigenvalues $\pm\frac{1}{2}f^2$, where f^2 can be expressed in terms of F by (3.16). To characterize the eigenvectors, all we need is an explicit for f in terms of F. From (3.15) and (3.17) we get

$$f = Fe^{-i\phi} = \frac{F(F \cdot F - F \wedge F)^{\frac{1}{2}}}{[(F \cdot F)^2 + |F \wedge F|^2]^{\frac{1}{2}}} \,. \tag{3.20}$$

This invariant of the field can be expressed in terms of electric and magnetic fields by substituting (3.10).

This method for solving the eigenvector problem should be compared with the conventional matrix method using the tensor form (3.14). The matrix solution worked out by Synge [11] requires many pages of text. Obviously, our method is much simpler and cleaner. There is a great lesson in mathematical design to be learned from this example. In contrast to matrix algebra, geometric algebra enables us to represent linear transformations and carry out computations with them without introducing a basis. To my knowledge it is the only mathematical system with this capability. Surely as a principle of good mathematical design we should endeavor to extend this capability as far as possible. Accordingly, I suggest that the mathematical community should aim to develop a complete theory of "geometric representations" for linear functions, that is, representations in terms of geometric algebra. The geometric representation theory for orthogonal functions is already well developed, and members of this audience are familiar with its advantages over matrix representations. The example we have just considered shows that there are comparable advantageq in the representation of symmetric functions. Ref. [2] carries the theory quite a bit further. But geometric representation theory is still not as well developed as matrix representation theory. Indeed, physicists are so conditioned to matrix algebra that most of them work with matrix representations of Clifford Algebra rather than directly with the Clifford Algebra itself. I submit that is would be better mathematical design to turn things around, to regard geometric algebra as more fundamental than matrix algebra and to derive matrix representations from geometric representations. In other words, matrix algebra should be subservient to geometric algebra, arising whenever it is convenient to introduce a basis and work wlth lists and arrays of numbers or linear functions.

Now let us turn to the third major component of electrodynamics the equation of motion for a charged particle. Let $x = x(\tau)$ be the world line of a particle parametrized by proper time τ. The invariant velocity of the particle is a timelike unit vector $v = \dot{x} = dx/d\tau$. For a test particle of rest mass m and charge q in an electromagnetic field F, the equation of motion can be written in the coordinate-free form

$$m\dot{v} = qF \cdot v \,. \tag{3.21}$$

Set $m = q = 1$ for convenience.

Equation (3.21) is not very different from the usual covariant tensor formulation, which can be obtained by taking its components with respect to the basis $\{\gamma^\mu\}$. But I want to show you how to solve it in a way which can't even be formulated in tensor analysis. First we note that it implies $d(v^2)/d\tau = 2\dot{v} \cdot v = 2(F \cdot v) \cdot v = 2F \cdot (v \wedge v) = 0$. Therefore, the condition $v^2 = 1$ is a constant of motion, and we can represent the change in v as the unfolding of a Lorentz transformation with spin representation $R = R(\tau)$, that is, we can write

$$v = R\gamma_0 \widetilde{R} \,, \tag{3.22}$$

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where R is an element of $\mathcal{R}_{1,3}^+$ for which $R\widetilde{R} = 1$. Then we can replace (3.21) by the simpler spinor equation of motion

$$\dot{R} = \tfrac{1}{2}FR \,. \tag{3.23}$$

For a uniform field F, this has the elementary solution

$$R = e^{F\tau/2}R_0 \,, \tag{3.24}$$

where R_0 is a constant determined by the initial conditions. This gives us an explicit expression for the proper time dependence of v when substituted into (3.22). However, the subsequent integration of $\dot{x} = v$ to get an explicit equation for the particle history is a litle tricky for arbitrary F. The solution is obtained in Ref. [2], which also works out the solutions when F is a plane wave or a Coulomb field.

The spinor R can be regarded as an integrating factor which we introduced with equations (3.22) and (3.23) merely to simplify the original equation of motion (3.21). However, there is reason to believe that something much deeper is involved. We can define a comoving frame $\{\mathbf{e}_\mu = \mathbf{e}_\mu(\tau)\}$ attached to the particle history by the equations

$$\mathbf{e}_\mu = R\gamma_\mu\widetilde{R} \tag{3.25}$$

for $u = 0, 1, 2, 3$. For $\mathbf{e}_0 = v$ this agrees with (3.22). Now here is the amazing part, if we regard the particle as an electron and identify $\mathbf{e}_3$ with the direction of the electron spin, then the spin precession predicted by (3.24) agrees exactly with the prediction of the Dirac theory, except for the tiny radiative corrections. Moreover, the solution of (3.23) for a plane wave is identical to the corresponding solution of the Dirac equation (known as the Volkov solution in the literature). The relation of the Coulomb solution of (3.23) to the Coulomb solution of the Dirac equation has not been investigated. But we have enough to assert that the simple "classical equation" (3.23) has an intimate relation to the Dirac theory. I will say more about this in my next lecture.

4. ALGEBRAIC STRUCTURES

Now let me return to the general theme of unifying mathematics. We have seen how real and complex analysis can be united in a general geometric function theory. The theory of linear geometric functions which we discussed later should be regarded as a branch of the general theory concerned with the simplest class of geometric functions, however, for reasons given in Ref. [2], it should be integrated with the theory of multilinear functions from the beginning. Reference [2] also shows how the theory of manifolds and differential geometry can be incorporated in geometric function theory.

There is one other branch of mathematics which should be integrated into the theory — Abstract Algebra. Like linear algebra it can be incorporated in the *theory of geometric representations*. Thus, an abstract product $a \circ b$ will be represented as a bilinear function $a \circ b = f(a, b)$ in the geometric algebra. There will be many such representations, but some will be optimal in an appropriate sense.

Why should one bother with geometric representations? Why not be satisfied with the pure abstract formulation? There are many reasons. Representations of the different

algebras in a common language should make it easier to establish relations among them and possibly uncover hidden structure. It should facilitate applications of the algebraic structures to fields, such as physics, which are already formulated in terms of geometric algebra. It will make the great computational power of geometric algebra available for integrating and analyzing algebraic structures. Best of all, it should be mathematically rich and interesting. To convince you of this, let me show you a geometric representation of the octonian product. We represent the octonians as vectors $a, b, \ldots$ in $\mathcal{R}^8$. Then, with the proper choice of a multivector P in R_8, the octonian productis given by the geometric function

$$a \circ b = \langle\, abP\,\rangle_1\,, \tag{4.1}$$

where the subscript means vector part. I will not deprive you of a chance to find P for yourself before it is revealed by Kwasniewski in his lecture. Let me only give you the hint that it is built out of idempotents of the algebra.

Of course the theory of geometric representations should be extended to embrace Lie groups and Lie algebras. A start has been made in Ref. [2]. I conjectured there that every Lie algebra is isomorphic to a bivector algebra, that is, an algebra of bivectors under the commutator product. Lawyer-physicist Tony Smith has proved that this conjecture is true by pointing to results already in the literature. However, the task remains to construct explicit bivector representations for all the Lie algebras. The task should not be difficult, as a good start has already been made.

I must stop here, because my time as well as my vision is limited. The other speakers will show the great richness of Clifford Algebras approached from any of the three major perspectives, the algebraic, the geometric, or the physical.

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