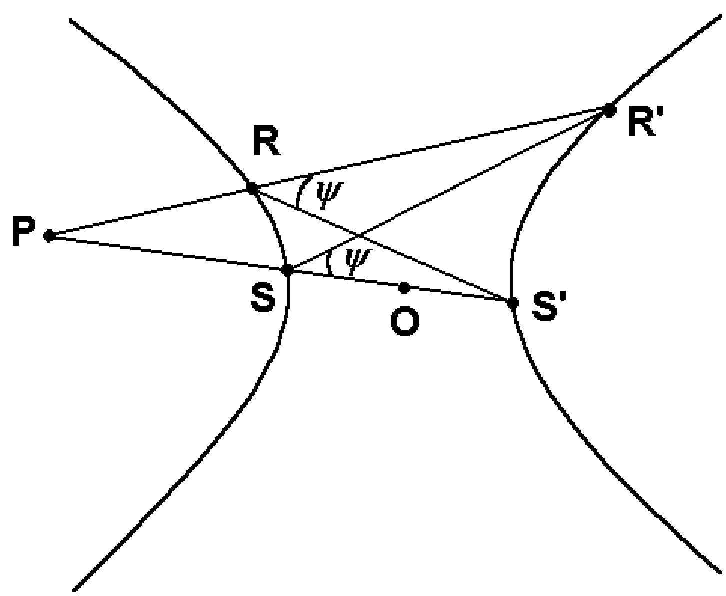
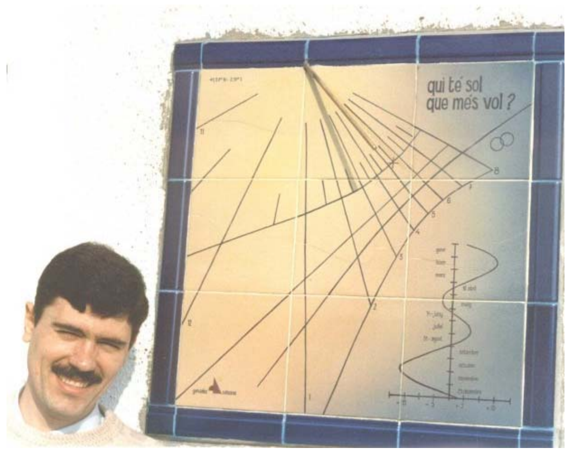
TREATISE OF PLANE GEOMETRY THROUGH GEOMETRIC ALGEBRA



Ramon González Calvet

TREATISE OF PLANE GEOMETRY THROUGH GEOMETRIC ALGEBRA

Ramon González Calvet



The geometric algebra, initially discovered by Hermann Grassmann (1809-1877) was reformulated by William Kingdon Clifford (1845-1879) through the synthesis of the Grassmann's extension theory and the quaternions of Sir William Rowan Hamilton (1805-1865). In this way the bases of the geometric algebra were established in the XIX century. Notwithstanding, due to the premature death of Clifford, the vector analysis –a remake of the quaternions by Josiah Willard Gibbs (1839-1903) and Oliver Heaviside (1850-1925)– became, after a long controversy, the geometric language of the XX century; the same vector analysis whose beauty attracted the attention of the author in a course on electromagnetism and led him -being still undergraduate- to read the Hamilton's *Elements of Quaternions*. Maxwell himself already applied the quaternions to the electromagnetic field. However the equations are not written so nicely as with vector analysis. In 1986 Ramon contacted Josep Manel Parra i Serra, teacher of theoretical physics at the Universitat de Barcelona, who acquainted him with the Clifford algebra. In the framework of the summer courses on geometric algebra which they have taught for graduates and teachers since 1994, the plan of writing some books on this subject appeared in a very natural manner, the first sample being the *Tractat de geometria plana mitjançant l'àlgebra geomètrica* (1996) now out of print. The good reception of the readers has encouraged the author to write the *Treatise of plane geometry through geometric algebra* (a very enlarged translation of the *Tractat*) and publish it at the Internet site http://campus.uab.es/~PC00018, writing it not only for mathematics students but also for any person interested in geometry. The plane geometry is a basic and easy step to enter into the Clifford-Grassmann geometric algebra, which will become the geometric language of the XXI century.

Dr. Ramon González Calvet (1964) is high school teacher of mathematics since 1987, fellow of the Societat Catalana de Matemàtiques (http://www-ma2.upc.es/~sxd/scma.htm) and also of the Societat Catalana de Gnomònica (http://www.gnomonica.org).

TREATISE OF PLANE GEOMETRY

THROUGH GEOMETRIC ALGEBRA

Dr. Ramon González Calvet

Mathematics Teacher
I.E.S. Pere Calders, Cerdanyola del Vallès

To my son Pere, born with the book.

© Ramon González Calvet (rgonzal1@teleline.es)
This is an electronic edition by the author at the Internet site http://campus.uab.es/~PC00018. All the rights reserved. Any electronic or paper copy cannot be reproduced without his permission. The readers are authorised to print the files only for his personal use. Send your comments or opinion about the book to ramon.gonzalezc@campus.uab.es .

ISBN: 84-699-3197-0

First Catalan edition: June 1996
First English edition: June 2000 to June 2001

PROLOGUE

The book I am so pleased to present represents a true innovation in the field of the mathematical didactics and, specifically, in the field of geometry. Based on the long neglected discoveries made by Grassmann, Hamilton and Clifford in the nineteenth century, it presents the geometry -the elementary geometry of the plane, the space, the spacetime- using the best algebraic tools designed specifically for this task, thus making the subject democratically available outside the narrow circle of individuals with the high visual imagination capabilities and the true mathematical insight which were required in the abandoned classical Euclidean tradition. The material exposed in the book offers a wide repertory of geometrical contents on which to base powerful, reasonable and up-to-date reintroductions of geometry to present-day high school students. This longed-for reintroductions may (or better should) take advantage of a combined use of symbolic computer programs and the cross disciplinary relationships with the physical sciences.

The proposed introduction of the geometric Clifford-Grassmann algebra in high school (or even before) follows rightly from a pedagogical principle exposed by William Kingdon Clifford (1845-1879) in his project of teaching geometry, in the University College of London, as a practical and empirical science as opposed to Cambridge Euclidean axiomatics: " ... for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child". Fellow of the Royal Society at the age of 29, Clifford also gave a set of *Lectures on Geometry* to a Class of Ladies at South Kengsinton and was deeply concerned in developing with MacMillan Company a series of **inexpensive** "very good elementary schoolbook of arithmetic, geometry, animals, plants, physics ...". Not foreign to this proposal are Felix Klein lectures to teachers collected in his book *Elementary mathematics from an advanced standpoint*[1] and the advice of Alfred North Whitehead saying that "the hardest task in mathematics is the study of the elements of algebra, and yet this stage must precede the comparative simplicity of the differential calculus" and that "the postponement of difficulty mis no safe clue for the maze of educational practice"[2].

Clearly enough, when the fate of pseudo-democratic educational reforms, disguised as a back to basic leitmotifs, has been answered by such an acute analysis by R. Noss and P. Dowling under the title *Mathematics in the National Curriculum: The Empty Set?*[3], the time may be ripen for a reappraisal of true pedagogical reforms based on a real knowledge, of substantive contents, relevant for each individual worldview construction. We believe that the introduction of the *vital or experiential* plane, space and space-time geometries along with its proper algebraic structures will be a substantial part of a successful (high) school scientific curricula. Knowing, telling, learning why the sign rule, or the complex numbers, or matrices are mathematical structures correlated to the human representation of the real world are worthy objectives in mass education projects. And this is possible today if we learn to stand upon the shoulders of giants such as Leibniz, Hamilton, Grassmann, Clifford, Einstein, Minkowski, etc. To this aim this book, offered and opened to suggestions to the whole world of concerned people, may be a modest but most valuable step towards these very good schoolbooks that constituted one of the cheerful Clifford's aims.

---

[1] Felix Klein, *Elementary mathematics from an advanced standpoint*. Dover (N. Y., 1924).
[2] A.N. Whitehead, *The aims of education*. MacMillan Company (1929), Mentor Books (N.Y., 1949).
[3] P. Dowling, R. Noss, eds., *Mathematics versus the National Curriculum: The Empty Set?*. The Falmer Press (London, 1990).

Finally, some words borrowed from Whitehead and Russell, that I am sure convey some of the deepest feelings, thoughts and critical concerns that Dr. Ramon González has had in mind while writing the book, and that fully justify a work that appears to be quite removed from today high school teaching, at least in Catalunya, our country.

> "Where attainable knowledge could have changed the issue, ignorance has the guilt of vice"[2].
> "The uncritical application of the principle of necessary antecedence of some subjects to others has, in the hands of dull people with a turn for organisation, produced in education the dryness of the Sahara"[2].
> "When one considers in its length and in its breadth the importance of this question of the education of a nation's young, the broken lives, the defeated hopes, the national failures, which result from the frivolous inertia with which it is treated, it is difficult to restrain within oneself a savage rage"[2].
> "A taste for mathematics, like a taste for music, can be generated in some people, but not in others. ... But I think that these could be much fewer than bad instruction makes them seem. Pupils who have not an unusually strong natural bent towards mathematics are led to hate the subject by two shortcomings on the part of their teachers. The first is that mathematics is not exhibited as the basis of all our scientific knowledge, both theoretical and practical: the pupil is convincingly shown that what we can understand of the world, and what we can do with machines, we can understand and do in virtue of mathematics. The second defect is that the difficulties are not approached gradually, as they should be, and are not minimised by being connected with easily apprehended central principles, so that the edifice of mathematics is made to look like a collection of detached hovels rather than a single temple embodying a unitary plan. It is especially in regard to this second defect that Clifford's book (*Common Sense of the Exact Sciences*) is valuable.(Russell)"[4].

An appreciation that Clifford himself had formulated, in his fundamental paper upon which the present book relies, relative to the *Ausdehnungslehre* of Grassmann, expressing "my conviction that its principles will exercise a vast influence upon the future of mathematical science".

Josep Manel Parra i Serra, June 2001

Departament de Física Fonamental
Universitat de Barcelona

[4] W. K. Clifford, *Common Sense of the Exact Sciences*. Alfred A. Knopf (1946), Dover (N.Y., 1955).

« On demande en second lieu, laquelle des deux qualités doit être préférée dans des *élémens*, de la facilité, ou de la rigour exacte. Je réponds que cette question suppose una chose fausse; elle suppose que la rigour exacte puisse exister sans la facilité & c'est le contraire; plus une déduction est rigoureause, plus elle est facile à entendre: car la rigueur consiste à reduire tout aux principes les plus simples. D'où il s'ensuit encore que la rigueur proprement dit entraîne nécessairement la méthode la plus naturelle & la plus directe. Plus les principles seront disposés dans l'ordre convenable, plus la déduction sera rigourease; ce n'est pas qu'absolument elle ne pût l'être si on suivonit une méthode plus composée, com a fait Euclide dans ses *élémens*: mais alors l'embarras de la marche feroit aisément sentir que cette rigueur précaire & forcée ne seroit qu'improprement telle. »[5]

["Secondly, one requests which of the two following qualities must be preferred within the *elements*, whether the easiness or the exact rigour. I answer that this question implies a falsehood; it implies that the exact rigour can exist without the easiness and it is the other way around; the more rigorous a deduction will be, the more easily it will be understood: because the rigour consists of reducing everything to the simplest principles. Whence follows that the properly called rigour implies necessarily the most natural and direct method. The more the principles will be arranged in the convenient order, the more rigorous the deduction will be; it does not mean that it cannot be rigorous at all if one follows a more composite method as Euclid made in his *elements*: but then the difficulty of the march will make us to feel that this precarious and forced rigour will only be an improper one."]

Jean le Rond D'Alembert (1717-1783)

[5] «Elémens des sciences» in *Encyclopédie, ou dictionaire raisonné des sciences, des arts et des métiers* (París, 1755).

PREFACE TO THE FIRST ENGLISH EDITION

The first edition of the *Treatise of Plane Geometry through Geometric Algebra* is a very enlarged translation of the first Catalan edition published in 1996. The good reception of the book (now out of print) encouraged me to translate it to the English language rewriting some chapters in order to make easier the reading, enlarging the others and adding those devoted to the non-Euclidean geometry.

The *geometric algebra* is the tool which allows to study and solve geometric problems through a simpler and more direct way than a purely geometric reasoning, that is, by means of the algebra of geometric quantities instead of *synthetic* geometry. In fact, the geometric algebra is the Clifford algebra generated by the Grassmann's outer product in a vector space, although for me, the geometric algebra is also the art of stating and solving geometric equations, which correspond to geometric problems, by isolating the unknown geometric quantity using the algebraic rules of the vectors operations (such as the associative, distributive and permutative properties). Following Peano[6]:

> "The geometric Calculus differs from the Cartesian Geometry in that whereas the latter operates analytically with coordinates, the former operates directly on the geometric entities".

Initially proposed by Leibniz[7] (*characteristica geometrica*) with the aim of finding an intrinsic language of the geometry, the geometric algebra was discovered and developed by Grassmann[8], Hamilton and Clifford during the XIX century. However, it did not become usual in the XX century ought to many circumstances but the *vector analysis* -a recasting of the Hamilton quaternions by Gibbs and Heaviside- was gradually accepted in physics. On the other hand, the geometry followed its own way aside from the vector analysis as Gibbs[9] pointed out:

> "And the growth in this century of the so-called synthetic as opposed to analytical geometry seems due to the fact that by the ordinary analysis geometers could not easily express, except in a cumbersome and unnatural manner, the sort of relations in which they were particularly interested"

---

[6] Giuseppe Peano, «Saggio di Calcolo geometrico». Translated in *Selected works of Giuseppe Peano*, 169 (see the bibliography).

[7] C. I. Gerhardt, *G. W. Leibniz. Mathematical Schriften* **V**, 141 and *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematiker*, 570.

[8] In 1844 a prize (45 gold ducats for 1846) was offered by the Fürstlich Jablonowski'schen Gessellschaft in Leipzig to whom was capable to develop the *characteristica geometrica* of Leibniz. Grassmann won this prize with the memoir *Geometric Analysis*, published by this society in 1847 with a foreword by August Ferdinand Möbius. Its contents are essentially those of *Die Ausdehnungslehre* (1844).

[9] Josiah Willard Gibbs, «On Multiple Algebra», reproduced in *Scientific papers of J.W. Gibbs*, **II**, 98.

The work of revision of the history and the sources (see J. M. Parra[10]) has allowed us to synthesise the contributions of the different authors and completely rebuild the evolution of the geometric algebra, removing the conceptual mistakes which led to the vector analysis. This preface has not enough extension to explain all the history[11], but one must remember something usually forgotten: during the XIX century several points of view over what should become the geometric algebra came into competition. The Gibbs' vector analysis was one of these being not the better. In fact, the geometric algebra is a field of knowledge where different formulations are possible as Peano showed:

> "Indeed these various methods of geometric calculus do not at all contradict one another. They are various parts of the same science, or rather various ways of presenting the same subject by several authors, each studying it independently of the others.
>
> It follows that geometric calculus, like any other method, is not a system of conventions but a system of truth. In the same way, the methods of indivisibles (Cavalieri), of infinitesimals (Leibniz) and of fluxions (Newton) are the same science, more or less perfected, explained under different forms."[12]

The geometric algebra owns some fundamental geometric facts which cannot be ignored at all and will be recognised to it, as Grassmann hoped:

> "For I remain completely confident that the labour which I have expanded on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remained unused for another seventeen years or even longer, without entering into the actual development of science, still the time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of truth, however small may be the region encompassed, can pass on without leaving

[10] Josep Manel Parra i Serra, «Geometric algebra versus numerical Cartesianism. The historical trend behind Clifford's algebra», in Brackx *et al*. ed., *Clifford Algebras and their Applications in Mathematical Physics*, 307-316, .

[11] A very complete reference is Michael J. Crowe, *A History of Vector Analysis. The Evolution of the Idea of a Vectorial System*.

[12] Giuseppe Peano, *op. cit.,* 168.

a trace; truth remains, even though the garment in which poor mortals clothe it may fall to dust."[13]

As any other aspect of the human life, the history of the geometric algebra was conditioned by many fortuitous events. While Grassmann deduced the *extension theory* from philosophic concepts unintelligible for authors such as Möbius and Gibbs, Hamilton identified vectors and bivectors -the starting point of the great tangle of vector analysis- using a heavy notation[14]. Clifford had found the correct algebraic structure[15] which integrated the systems of Hamilton and Grassmann. However due to the premature death of Clifford in 1879, his opinion was not taken into account[16] and a long epistolary war was carried out by the quaternionists (specially Tait) against the defenders of the vector analysis, created by Gibbs[17], who did not recognise to be influenced by Grassmann and Hamilton:

> "At all events, I saw that the methods which I was using, while nearly those of Hamilton, were almost exactly those of Grassmann. I procured the two Ed. of the Ausdehnungslehre but I cannot say that I found them easy reading. In fact I have never had the perseverance to get through with either of them, and have perhaps got more ideas from his miscellaneous memoirs than from those works.
>
> I am not however conscious that Grassmann's writings exerted any particular influence on my Vector Analysis, although I was glad enough in the introductory paragraph to shelter myself behind one or two distinguished names [Grassmann and Clifford] in making changes of notation which I felt would be distasteful to quaternionists. In fact if you read that pamphlet carefully you will see that it all follows with the inexorable logic of algebra from the problem which I had set myself long before my acquaintance with Grassmann.
>
> I have no doubt that you consider, as I do, the methods of Grassmann to be superior to those of Hamilton. It thus seemed to me that it might [be] interesting to you to know how commencing with some knowledge of Hamilton's method and influenced simply by a desire to obtain the simplest algebra for the expression of the relations of Geom. Phys. etc. I was led essentially to Grassmann's algebra of vectors, independently of any influence from him or any one else."[18]

---

[13] Hermann Gunther Grassmann. Preface to the second edition of *Die Ausdehnungslehre* (1861). The first edition was published on 1844, hence the "seventeen years". Translated in Crowe, *op. cit.* p. 89.

[14] The *Lectures on Quaternions* was published in 1853, and the *Elements of Quaternions* posthumously in 1866.

[15] William Kingdon Clifford left us his synthesis in «Applications of Grassmann's Extensive Algebra».

[16] See «On the Classification of Geometric Algebras», unfinished paper whose abstract was communicated to the London Mathematical Society on March 10th, 1876.

[17] The first *Vector Analysis* was a private edition of 1881.

[18] Draft of a letter sent by Josiah Willard Gibbs to Victor Schlegel (1888). Reproduced by Crowe, *op. cit*. p. 153.

Before its beginning the controversy was already superfluous[19]. Notwithstanding the epistolary war continued for twelve years.

The vector analysis is a provisional solution[20] (which spent all the XX century!) adopted by everybody ought to its easiness and practical notation but having many troubles when being applied to three-dimensional geometry and unable to be generalised to the Minkowski's four-dimensional space. On the other hand, the geometric algebra is, by its own nature, valid in any dimension and it offers the necessary resources for the study and research in geometry as I show in this book.

The reader will see that the theoretical explanations have been completed with problems in each chapter, although this splitting is somewhat fictitious because the problems are demonstrations of geometric facts, being one of the most interesting aspects of the geometric algebra and a proof of its power. The usual numeric problems, which our pupils like, can be easily outlined by the teacher, because the geometric algebra always yields an immediate expression with coordinates.

I'm indebted to professor Josep Manel Parra for encouraging me to write this book, for the dialectic interchange of ideas and for the bibliographic support. In the framework of the summer courses on geometric algebra for teachers that we taught during the years 1994-1997 in the *Escola d'estiu de secundària* organised by the Col·legi Oficial de Doctors i Llicenciats en Filosofia i Lletres i en Ciències de Catalunya, the project of some books on this subject appeared in a natural manner. The first book devoted to two dimensions already lies on your hands and will probably be followed by other books on the algebra and geometry of the three and four dimensions.

Finally I also acknowledge the suggestions received from some readers.

Ramon González Calvet

Cerdanyola del Vallès, June 2001

---

19 See Alfred M. Bork «"Vectors versus quaternions"—The letters in *Nature*».

20 The vector analysis bases on the duality of the geometric algebra of the three-dimensional space: the fact that the orientation of lines and planes is determined by three numeric components in both cases. However in the four-dimensional time-space the same orientations are respectively determined by four and six numbers.

CONTENTS

FIRST PART: THE VECTOR PLANE AND THE COMPLEX NUMBERS

Points and vectors are the main elements of the plane geometry. A *point* is conceived (but not defined) as a geometric element without extension, infinitely small, that has position and is located at a certain place on the plane. A *vector* is defined as an oriented segment, that is, a piece of a straight line having length and direction. A vector has no position and can be translated anywhere. Usually it is called a *free* vector. If we place the end of a vector at a point, then its head determines another point, so that a vector represents the translation from the first point to the second one.

Taking into account the distinction between points and vectors, the part of the book devoted to the Euclidean geometry has been divided in two parts. In the first one the vectors and their algebraic properties are studied, which is enough for many scientific and engineering branches. In the second part the points are introduced and then the affine geometry is studied.

All the elements of the geometric algebra (scalars, vectors, bivectors, complex numbers) are denoted with lowercase Latin characters and the angles with Greek characters. The capital Latin characters will denote points on the plane. As you will see, the geometric product is not commutative, so that fractions can only be written for real and complex numbers. Since the geometric product is associative, the inverse of a certain element at the left and at the right is the same, that is, there is a unique inverse for each element of the algebra, which is indicated by the superscript $^{-1}$. Also due to the associative property, all the factors in a product are written without parenthesis. In order to make easy the reading I have not numerated theorems, corollaries nor equations. When a definition is introduced, the definite element is marked with italic characters, which allows to direct attention and helps to find again the definition.

1. THE VECTORS AND THEIR OPERATIONS

A vector is an oriented segment, having length and direction but no position, that is, it can be placed anywhere without changing its orientation. The vectors can represent many physical magnitudes such as a force, a celerity, and also geometric magnitudes such as a translation.

Two algebraic operations for vectors are defined, the addition and the product, that generalise the addition and product of the real numbers.

Vector addition

The *addition* of two vectors $u + v$ is defined as the vector going from the end of the vector u to the head of v when the head of u contacts the end of v (upper triangle in the figure 1.1). Making the construction for $v + u$, that is, placing the end of u at the head of v (lower triangle in the figure 1.1) we see that the addition vector is the same. Therefore, the vector addition has the commutative property:

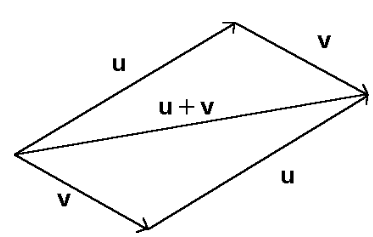


Figure 1.1

$$u + v = v + u$$

and the parallelogram rule follows: the addition of two vectors is the diagonal of the parallelogram formed by both vectors. The associative property follows from this definition because $(u+v)+w$ or $u+(v+w)$ is the vector closing the polygon formed by the three vectors as shown in the figure 1.2.

The neutral element of the vector addition is the *null vector*, which has zero length. Hence the *opposite vector* of u is defined as the vector $-u$ with the same orientation but opposite direction, which added to the initial vector gives the null vector:

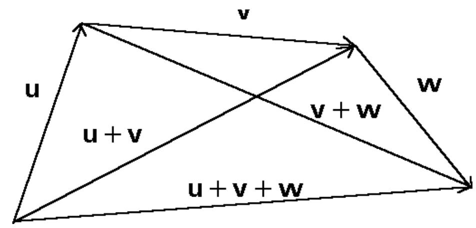


Figure 1.2

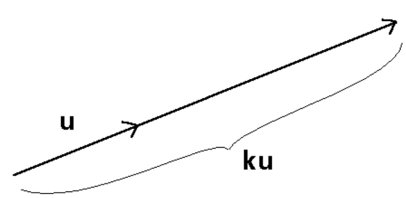
$$u + (-u) = 0$$

Product of a vector and a real number

One defines the product of a vector and a real number (or *scalar*) k, as a vector with the same direction but with a length increased k times (figure 1.3). If the real number is negative, then the direction is the opposite. The geometric definition implies the commutative property:

$$k\, u = u\, k$$

Figure 1.3



Two vectors u, v with the same direction are *proportional* because there is always a real number k such that $v = k\, u$, that is, k is the quotient of both vectors:

$$k = u^{-1}\, v = v\, u^{-1}$$

Two vectors with different directions are said to be *linearly independent*.

Product of two vectors

The product of two vectors will be called the *geometric product* in order to be distinguished from other vector products currently used. Nevertheless I hope that these other products will play a secondary role when the geometric product becomes the most used, a near event which this book will forward. At that time, the adjective «geometric» will not be necessary.

The following properties are demanded to the geometric product of two vectors:

1) To be distributive with regard to the vector addition:

$$u\,(\,v + w\,) = u\,v + u\,w$$

2) The square of a vector must be equal to the square of its length. By definition, the length (or *modulus*) of a vector is a positive number and it is noted by $|\,u\,|$:

$$u^2 = |\,u\,|^2$$

3) The mixed associative property must exist between the product of vectors and the product of a vector and a real number.

$$k\,(\,u\,v\,) = (\,k\,u\,)\,v = \;k\,u\,v$$

$$k\,(\,l\,u\,) = (\,k\,l\,)\,u = \;k\,l\,u$$

where k, l are real numbers and u, v vectors. Therefore, parenthesis are not needed.

These properties allows us to deduce the product. Let us suppose that c is the addition of two vectors a, b and calculate its square applying the distributive property:

$$c = a + b$$

$$c^2 = (\,a + b\,)^2 = (\,a + b\,)\,(\,a + b\,) = a^2 + a\,b + b\,a + b^2$$

We have to preserve the order of the factors because we do not know whether the product is commutative or not.

If a and b are orthogonal vectors, the Pythagorean theorem applies and then:

$$a \perp b \quad \Rightarrow \quad c^2 = a^2 + b^2 \quad \Rightarrow \quad a\,b + b\,a = 0 \quad \Rightarrow \quad a\,b = -\,b\,a$$

That is, <u>the product of two perpendicular vectors is anticommutative</u>.

If a and b are proportional vectors then:

$$a \parallel b \quad \Rightarrow \quad b = k\,a, \;\; k \text{ real} \quad \Rightarrow \quad a\,b = a\,k\,a = k\,a\,a = b\,a$$

because of the commutative and mixed associative properties of the product of a vector and a real number. Therefore <u>the product of two proportional vectors is commutative</u>. If c is the addition of two vectors a, b with the same direction, we have:

$$|\,c\,| = |\,a\,| + |\,b\,|$$

$$c^2 = a^2 + b^2 + 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = |\,a\,|\,|\,b\,| \qquad \text{angle}(a,\,b) = 0$$

But if the vectors have opposite directions:

$$|\,c\,| = |\,a\,| - |\,b\,|$$

$$c^2 = a^2 + b^2 - 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = -\,|\,a\,|\,|\,b\,| \qquad \text{angle}(a,\, b) = \pi$$

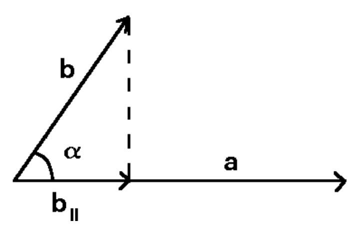
How is the product of two vectors with any directions? Due to the distributive property the product is resolved into one product by the proportional component $b_{||}$ and another by the orthogonal component $b_{\perp}$:

$$a\,b = a\,(\,b_{||} + b_{\perp}\,) = a\,b_{||} + a\,b_{\perp}$$

The product of one vector by the proportional component of the other is called the *inner product* (also *scalar product*) and noted by a point · (figure 1.4). Taking into account that the projection of b onto a is proportional to the cosine of the angle between both vectors, one finds:

$$a \cdot b = a\,b_{||} = |\,a\,|\,|\,b\,| \cos \alpha$$

Figure 1.4



The inner product is always a real number. For example, the work made by a force acting on a body is the inner product of the force and the walked space. Since the commutative property has been deduced for the product of vectors with the same direction, it follows also for the inner product:

$$a \cdot b = b \cdot a$$

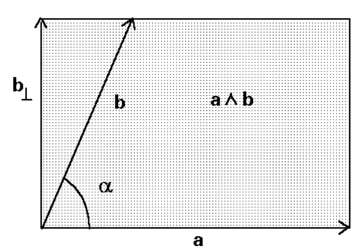
The product of one vector by the orthogonal component of the other is called the *outer product* (also *exterior product*) and it is noted with the symbol ∧ :

$$a \wedge b = a\,b_{\perp}$$

The outer product represents the area of the parallelogram formed by both vectors (figure 1.5):

$$\left|\; a \wedge b \;\right| = \left|\; a\,b_{\perp} \;\right| = \left|a\right|\,\left|b\right|\,\left|\sin \alpha\right|$$

Figure 1.5



Since the outer product is a product of orthogonal vectors, it is anticommutative:

$$a \wedge b = -\,b \wedge a$$

Some example of physical magnitudes which are outer products are the angular momentum, the torque, etc.

When two vectors are permuted, the sign of the oriented angle is changed. Then the cosine remains equal while the sine changes the sign. Because of this, the inner product is commutative while the outer product is anticommutative. Now, we can rewrite the geometric product as the sum of both products:

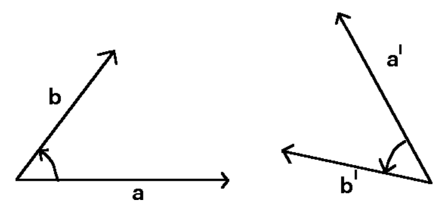
$$a\,b = a \cdot b + a \wedge b$$

From here, the inner and outer products can be written using the geometric product:

$$a \cdot b = \frac{a\,b + b\,a}{2}$$

$$a \wedge b = \frac{a\,b - b\,a}{2}$$

Figure 1.6



In conclusion, the geometric product of two proportional vectors is commutative whereas that of two orthogonal vectors is anticommutative, just for the pure cases of outer and inner products. The outer, inner and geometric products of two vectors only depend upon the moduli of the vectors and the angle between them. When both vectors are rotated preserving the angle that they form, the products are also preserved (figure 1.6).

How is the absolute value of the product of two vectors? Since the inner and outer product are linearly independent and orthogonal magnitudes, the modulus of the geometric product must be calculated through a generalisation of the Pythagorean theorem:

$$a\,b = a \cdot b + a \wedge b \quad \Rightarrow \quad |\,a\,b\,|^2 = |a \cdot b|^2 + |a \wedge b|^2$$

$$|\,a\,b\,|^2 = |\,a\,|^2\,|\,b\,|^2\,(\cos^2\alpha + \sin^2\alpha) = |\,a\,|^2\,|\,b\,|^2$$

That is, the modulus of the geometric product is the product of the modulus of each vector:

$$|\,a\,b\,| = |\,a\,|\,|\,b\,|$$

Product of three vectors: associative property

It is demanded as the fourth property that the product of three vectors be associative:

4) $u\,(\,v\,w\,) = (\,u\,v\,)\,w = u\,v\,w$

Hence we can remove parenthesis in multiple products and with the foregoing properties we can deduce how the product operates upon vectors.

We wish to multiply a vector *a* by a product of two vectors *b, c*. We ignore the result of the product of three vectors with different orientations except when two adjacent factors are proportional. We have seen that the product of two vectors depends only on the angle between them. Therefore the parallelogram formed by *b* and *c* can be rotated until b has, in the new orientation, the same direction as a. If b' and c' are the vectors b and c with the new orientation (figure 1.7) then:

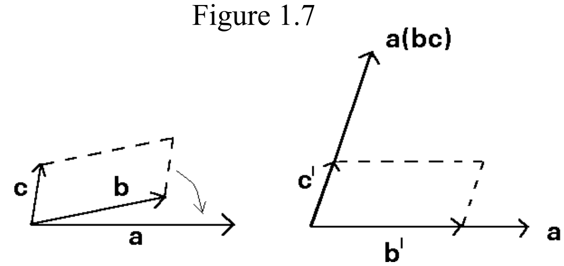
$$b\,c = b'\,c'$$

$$a\,(\,b\,c\,) = a\,(\,b'\;c'\,)$$

and by the associative property:

$$a\,(\,b\,c\,) = (\,a\,b'\,)\,c'$$

Figure 1.7



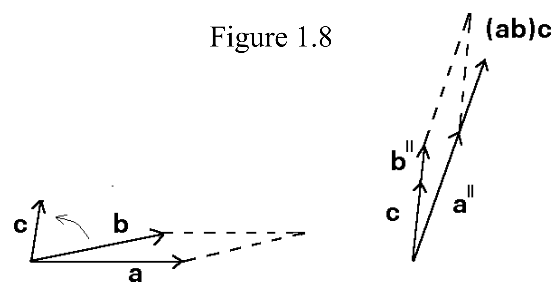
Since a and b' have the same direction, $a\;b' = |a|\,|b|$ is a real number and the triple product is a vector with the direction of c' whose length is increased by this amount:

$$a\,(\,b\,c\,) = |a|\,|b|\;c'$$

It follows that the modulus of the product of three vectors is the product of their moduli:

$$|a\,b\,c| = |a|\,|b|\,|c|$$

Figure 1.8



On the other hand, a can be firstly multiplied by b, and after this we can rotate the parallelogram formed by both vectors until b has, in the new orientation, the same direction as c (figure 1.8). Then:

$$(\,a\,b\,)\,c \;=\; a''(\;b''\,c\,) = a''\;|b|\,|c|$$

Although the geometric construction differs from the foregoing one, the figures clearly show that the triple product yields the same vector, as expected from the associative property.

$$(\,a\,b\,)\,c \;= a''\;|b|\,|c| \;=\; |c|\,|b|\;a'' = c\,b''\,a'' = c\,(\,b\,a\,)$$

That is, the triple product has the property:

$$a\,b\,c = c\,b\,a$$

which I call the *permutative* property: every vector can be permuted with a vector located two positions farther in a product, although it does not commute with the neighbouring vectors. The permutative property implies that any pair of vectors in a product separated by an odd number of vectors can be permuted. For example:

$$a\,b\,c\,d = a\,d\,c\,b = c\,d\,a\,b = c\,b\,a\,d$$

The permutative property is characteristic of the plane and it is also valid for the space whenever the three vectors are coplanar. This property is related with the fact that the product of complex numbers is commutative.

Product of four vectors

The product of four vectors can be deduced from the former reasoning. In order to multiply two pair of vectors, rotate the parallelogram formed by *a* and *b* until *b'* has the direction of *c*. Then the product is the parallelogram formed by *a'* and *d* but increased by the modulus of *b* and *c*:

$$a\, b\, c\, d = a'\, b'\, c\, d = a' \,|\, b \,|\, |\, c \,|\, d = |\, b \,|\, |\, c \,|\, a'\, d$$

Now let us see the special case when $a = c$ and $b = d$. If both vectors *a, b* have the same direction, the square of their product is a positive real number:

$$a \parallel b \qquad (a\, b)^2 = a^2\, b^2 > 0$$

If both vectors are perpendicular, we must rotate the parallelogram through $\pi/2$ until *b'* has the same direction as *a* (figure 1.9). Then *a'* and *b* are proportional but having opposite signs. Therefore, the square of a product of two orthogonal vectors is always negative:

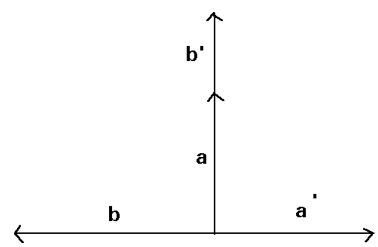


Figure 1.9

$$a \perp b \qquad (a\, b)^2 = a'\, b'\, a\, b = a' |\, b \,|\, |\, a \,|\, b = -a^2\, b^2 < 0$$

Likewise, the square of an outer product of any two vectors is also negative.

Inverse and quotient of two vectors

The *inverse* of a vector *a* is that vector whose multiplication by *a* gives the unity. Only the vectors which are proportional have a real product. Hence the inverse vector has the same direction and inverse modulus:

$$a^{-1} = a\,|\, a \,|^{-2} \qquad \Rightarrow \qquad a^{-1}\, a = a\, a^{-1} = 1$$

The quotient of vectors is a product for an inverse vector, which depends on the order of the factors because the product is not commutative:

$$a^{-1}\, b \neq b\, a^{-1}$$

Obviously the quotient of proportional vectors with the same direction and sense is equal to the quotient of their moduli. When the vectors have different directions, their quotient can be represented by a parallelogram, which allows to extend the concept of vector proportionality. We say that a is proportional to c as b is to d when their moduli are proportional and the angle between a and c is equal to the angle between b and d[1]:

$$a\,c^{-1} = b\,d^{-1} \quad \Leftrightarrow \quad |a|\,|c|^{-1} = |b|\,|d|^{-1} \quad \text{and} \quad \alpha(a, c) = \alpha(b, d)$$

Then the parallelogram formed by a and b is similar to that formed by c and d, being $\alpha(a, c)$ the angle of rotation from the first to the second one.

The inverse of a product of several vectors is the product of the inverses with the exchanged order, as can be easily seen from the associative property:

$$(\,a\,b\,c\,)^{-1} = c^{-1}\,b^{-1}\,a^{-1}$$

Hierarchy of algebraic operations

Like the algebra of real numbers, and in order to simplify the algebraic notation, I shall use the following hierarchy for the vector operations explained above:

1) The parenthesis, whose content will be firstly operated.
2) The power with any exponent (square, inverse, etc.).
3) The outer and inner product, which have the same hierarchy level but must be operated before the geometric product.
4) The geometric product.
5) The addition.

As an example, some algebraic expressions are given with the simplified expression at the left hand and its meaning using parenthesis at the right hand:

$$a \wedge b\; c \wedge d = (\,a \wedge b\,)\,(\,c \wedge d\,)$$

$$a^2\, b \wedge c + 3 = (\,(\,a^2\,)\,(\,b \wedge c\,)\,) + 3$$

$$a + b \cdot c\; d\; e = a + (\,(\,b \cdot c\,)\, d\, e\,)$$

Geometric algebra of the vectorial plane

The set of all the vectors on the plane together with the operations of vector addition and product of vectors by real numbers is a two-dimensional space usually called the *vector plane* V_2. The geometric product generates new elements (the complex numbers) not included in the vector plane. So, the geometric (or Clifford) algebra of a vectorial space is defined as the set of all the elements generated by products of vectors, for which the geometric product is an inner operation. The geometric algebra of the Euclidean vector plane is usually noted as $Cl_{2,0}(\mathbf{R})$ or simply as Cl_2. Making a parallelism with probability, the sample space is the set of elemental results of a certain

[1] William Rowan Hamilton defined the quaternions as quotients of two vectors in the way that similar parallelograms located at the same plane in the space represent the same quaternion (*Elements of Quaternions*, posthumously edited in 1866, Chelsea Publishers 1969, vol I, see p. 113 and fig. 34). In the vectorial plane a quaternion is reduced to a complex number. The quaternions were discovered by Hamilton (October 16th, 1843) before the geometric product by Clifford (1878).

random experiment. From the sample space Ω, the union $\cup$ and intersection $\cap$ generate the Boole algebra $A(\Omega)$, which includes all the possible events. In the same manner, the addition and geometric product generate the geometric algebra of the vectorial space. Then both sample and vectorial space play similar roles as generators of the Boole and geometric (Clifford) algebras respectively.

Exercises

1.1 Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides. Think about the sides as vectors.

1.2 Prove the following identity:

$$(a \cdot b)^2 - (a \wedge b)^2 = a^2\, b^2$$

1.3 Prove that: $a \wedge b\; c \wedge d = a \cdot (b\; c \wedge d) = (a \wedge b\; c) \cdot d$

1.4 Prove that: $a \wedge b\; c \wedge d + a \wedge c\; d \wedge b + a \wedge d\; b \wedge c = 0$

1.5 Prove the permutative property resolving b and c into the components which are proportional and orthogonal to the vector a.

1.6 Prove the Heron's formula for the area of the triangle:

$$A = \sqrt{s\,(s - |a|)\,(s - |b|)\,(s - |c|)}$$

where a, b and c are the sides and s the semiperimeter:

$$s = \frac{|a| + |b| + |c|}{2}$$

2. A BASE OF VECTORS FOR THE PLANE

Linear combination of two vectors

Every vector w on the plane is always a linear combination of two independent vectors u and v:

$$w = a\,u + b\,v \qquad a,\, b \text{ real}$$

Because of this, the plane has dimension equal to 2. In order to calculate the coefficients of linear combination a and b, we multiply w by u and v at both sides and subtract the results:

$$u\,w = a\,u^2 + b\,u\,v \quad \text{and} \quad w\,u = a\,u^2 + b\,v\,u \quad \Rightarrow \quad u\,w - w\,u = b\,(\,u\,v - v\,u\,)$$

$$v\,w = a\,v\,u + b\,v^2 \quad \text{and} \quad w\,v = a\,u\,v + b\,v^2 \quad \Rightarrow \quad w\,v - v\,w = a\,(\,u\,v - v\,u\,)$$

to obtain:

$$a = \frac{w \wedge v}{u \wedge v} \qquad\qquad b = \frac{u \wedge w}{u \wedge v}$$

The resolution of a vector as a linear combination of two independent vectors is a very frequent operation and also the foundation of the coordinates method.

Base and components

Any set of two independent vectors $\{e_1\,,\,e_2\}$ can be taken as a *base* of the vector plane. Every vector u can be written as linear combination of the base vectors:

$$u = u_1\,e_1 + u_2\,e_2$$

The coefficients of this linear combination u_1, u_2 are the *components* of the vector in this base. Then a vector will be represented as a pair of components:

$$u = (\,u_1,\,u_2\,)$$

The components depend on the base, so that a change of base leads to a change of the components of the given vector.

We must only add components to add vectors:

$$v = (\,v_1,\,v_2\,)$$

$$u + v = (\,u_1 + v_1\,,\,u_2 + v_2)$$

The expression of the geometric product with components is obtained by means of the distributive property:

$$u\, v = (\, u_1\, e_1 + u_2\, e_2\,)\, (\, v_1\, e_1 + v_2\, e_2\,) = u_1\, v_1\, e_1^{\,2} + u_2\, v_2\, e_2^{\,2} + u_1\, v_2\, e_1\, e_2 + u_2\, v_1\, e_2\, e_1$$

$$u\, v = u_1\, v_1\, |\, e_1\,|^2 + u_2\, v_2\, |\, e_2\,|^2 + (\, u_1\, v_2 + u_2\, v_1\,)\, e_1 \cdot e_2 + (\, u_1\, v_2 - u_2\, v_1\,)\, e_1 \wedge e_2$$

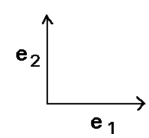
Hence the expression of the square of the vector modulus written in components is:

$$|\, u\,|^2 = u^2 = u_1^{\,2}\, |\, e_1\,|^2 + u_2^{\,2}\, |\, e_2\,|^2 + 2\, u_1\, u_2\, e_1 \cdot e_2$$

Orthonormal bases

Any base is valid to describe vectors using components, although the orthonormal bases, for which both e_1 and e_2 are unitary and perpendicular (such as the canonical base shown in the figure 2.1), are the more convenient and suitable:

Figure 2.1



$$e_1 \perp e_2 \qquad\qquad |\, e_1\,| = |\, e_2\,| = 1$$

For every orthonormal base :

$$e_1^{\,2} = e_2^{\,2} = 1 \qquad\qquad e_1\, e_2 = -\, e_2\, e_1$$

The product $e_1\, e_2$ represents a square of unity area. The square power of this product is equal to -1:

$$(\, e_1\, e_2\,)^2 = e_1\, e_2\, e_1\, e_2 = -\, e_1\, e_1\, e_2\, e_2 = -1$$

For an orthonormal base, the geometric product of two vectors becomes:

$$u\, v = u_1\, v_1 + u_2\, v_2 + (\, u_1\, v_2 - u_2\, v_1\,)\, e_1\, e_2$$

Note that the first and second summands are real while the third is an area. Therefore it follows that they are respectively the inner and outer products:

$$u \cdot v = u_1\, v_1 + u_2\, v_2 \qquad\qquad u \wedge v = (\, u_1\, v_2 - u_2\, v_1\,)\, e_1\, e_2$$

Also, the modulus of a vector is calculated from the self inner product:

$$|\, u\,|^{\,2} = u_1^{\,2} + u_2^{\,2}$$

Applications of the formulae for the products

The first application is the calculation of the angle between two vectors:

$$\cos\alpha = \frac{u_1 v_1 + u_2 v_2}{|\, u\,|\,|\, v\,|} \qquad\qquad \sin\alpha = \frac{u_1 v_2 - u_2 v_1}{|\, u\,|\,|\, v\,|}$$

The values of sine and cosine determine a unique angle α in the range $0<\alpha<2\pi$. The angle between two vectors is a sensed magnitude having positive sign if it is counterclockwise and negative sign if it is clockwise. Thus this angle depends on the order of the vectors in the outer (and geometric) product. For example, let us consider the vectors (figure 2.2) u and v:

$$u = (-2, 2) \qquad |u| = 2\sqrt{2} \qquad\qquad v = (4, 3) \qquad |v| = 5$$

$$\cos\alpha(u,v) = \cos\alpha(v,u) = -\frac{1}{5\sqrt{2}} \qquad\qquad \sin\alpha(u,v) = -\sin\alpha(v,u) = -\frac{7}{5\sqrt{2}}$$

$$\alpha(u, v) = 4.5705... \qquad\qquad \alpha(v, u) = 1.7127...$$

Also we may take the angle $\alpha(u, v) = 4.5705...-2\pi = -1.7127...$ The angle so obtained is always that going from the first to the second vector, being unique within a period.

Other application of the outer product is the calculus of areas. Using the expression with components, the area (considered as a positive real number) of the parallelogram formed by u and v is:

$$A = |u \wedge v| = |u_1 v_2 - u_2 v_1| = 14$$

When calculating the area of any triangle we must only divide the outer product of any two sides by 2.

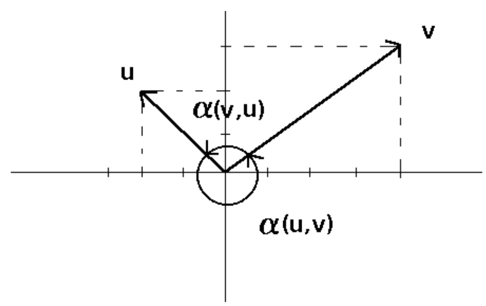


Figura 2.2

Exercises

2.1 Let (u_1, u_2) and (v_1, v_2) be the components of the vectors u and v in the canonical base. Prove geometrically that the area of the parallelogram formed by both vectors is the modulus of the outer product $|u \wedge v| = u_1 v_2 - u_2 v_1$.

2.2 Calculate the area of the triangle whose sides are the vectors (3, 5), (–2, –3) and their addition (1, 2).

2.3 Prove the permutative property using components: $a\,b\,c = c\,b\,a$.

2.4 Calculate the angle between the vectors $u = 2\,e_1 + 3\,e_2$ and $v = -3\,e_1 + 4\,e_2$ in the canonical base.

2.5 Consider a base where e_1 has modulus 1, e_2 has modulus 2 and the angle between both vectors is $\pi/3$. Calculate the angle between $u = 2\,e_1 + 3\,e_2$ and $v = -3\,e_1 + 4\,e_2$.

2.6 In the canonical base $v = (3, -5)$. Calculate the components of this vector in a new base $\{u_1, u_2\}$ if $u_1 = (2, -1)$ and $u_2 = (5, -3)$.

3. THE COMPLEX NUMBERS

Subalgebra of the complex numbers

If $\{e_1, e_2\}$ is the canonical base of the vector plane V_2, its *geometric algebra* is defined as the vector space generated by the elements $\{1, e_1, e_2, e_1e_2\}$ together with the geometric product, so that the geometric algebra Cl_2 has dimension four. The unitary area e_1e_2 is usually noted as e_{12}. Due to the associative character of the geometric product, the geometric algebra is an associative algebra with identity. The complete table for the geometric product is the following:

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

Note that the subset of elements containing only real numbers and areas is closed for the product:

	1	e_{12}
1	1	e_{12}
e_{12}	e_{12}	-1

This is the *subalgebra of complex numbers*. e_{12} , the *imaginary unity,* is usually noted as i. They are called *complex numbers* because their product is commutative like for the real numbers.

Binomial, polar and trigonometric form of a complex number

Every complex number z written in the *binomial* form is:

$$z = a + b\, e_{12} \qquad a, b \ \text{ real}$$

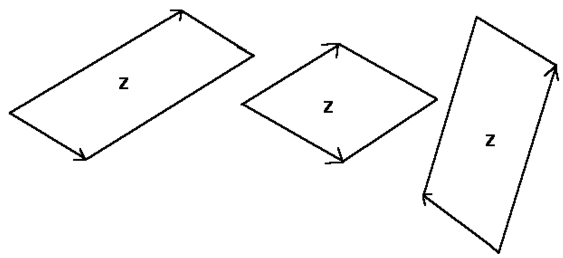
where a and b are the real and imaginary components respectively. The modulus of a complex number is calculated in the same way as the modulus of any element of the geometric algebra by means of the Pythagorean theorem:

$$|z|^2 = |a + b\, e_{12}|^2 = a^2 + b^2$$

Since every complex number can be written as a product of two vectors u and v forming a certain angle α:

$z = u\, v = |u|\ |v|\ (\cos \alpha + e_{12} \sin \alpha)$

Figure 3.1



we may represent a complex number as a parallelogram with sides being the vectors u and v. But there are infinite pairs of vectors u' and v' whose product is the complex z provided that:

$$|u|\ |v| = |u'|\ |v'| \quad \text{and} \quad \alpha = \alpha'$$

All the parallelograms having the same area and obliquity represent a given complex (they are equivalent) independently of the length and orientation of one side.(figure 3.1). The *trigonometric* and *polar* forms of a complex number z specifies its modulus $|z|$ and argument α :

$$z = |z|\ (\cos \alpha + e_{12} \sin \alpha) = |z|_{\ \alpha}$$

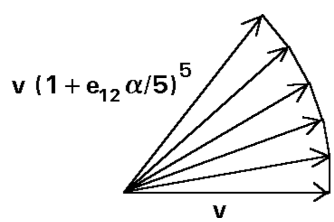
A complex number can be written using the exponential function, but firstly we must prove the Euler's identity:

$$\exp(\alpha\, e_{12}) = \cos \alpha + e_{12} \sin \alpha \qquad \alpha \text{ real}$$

The exponential of an imaginary number is defined in the same manner as for a real number:

$$\exp(\alpha\, e_{12}) = \lim_{n \to \infty} \left(1 + \frac{\alpha\, e_{12}}{n}\right)^n$$

Figure 3.2



As shown in the figure 3.2 (for $n = 5$), the limit is a power of n rotations with angle α/n or equivalently a rotation of angle α.

Now a complex number written in *exponential* form is:

$$z = |z|\ \exp(\alpha\, e_{12})$$

Algebraic operations with complex numbers

Each algebraic operation is more easily calculated in a form than in another according to the following scheme:

addition / subtraction	↔	binomial form
product / quotient	↔	binomial or polar form
powers / roots	↔	polar form

The binomial form is suitable for the addition because both real components must be added and also the imaginary ones, e.g.:

$$z = 3 + 4\, e_{12} \qquad\qquad t = 2 - 5\, e_{12}$$

$$z + t = 3 + 4\, e_{12} + 2 - 5\, e_{12} = 5 - e_{12}$$

If the complex numbers are written in another form, they must be converted to the binomial form, e.g.

$$z = 2_{3\pi/4} \qquad t = 4_{\pi/6}$$

$$z + t = 2_{3\pi/4} + 4_{\pi/6} = 2\left(\cos\frac{3\pi}{4} + e_{12}\,\sin\frac{3\pi}{4}\right) + 4\left(\cos\frac{\pi}{6} + e_{12}\,\sin\frac{\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} + e_{12}\,\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{3}}{2} + e_{12}\,\frac{1}{2}\right) = -\sqrt{2} + 2\sqrt{3} + e_{12}\left(\sqrt{2} + 2\right)$$

In order to write the result in polar form, the modulus must be calculated:

$$|\, z + t\,|^2 = (-\sqrt{2} + 2\sqrt{3}\,)^2 + (\sqrt{2} + 2\,)^2 = 20 + 4\,(\sqrt{2} - \sqrt{6}\,)$$

$$|\, z + t\,| = 2\sqrt{5 + \sqrt{2} - \sqrt{6}} = 3.9823...$$

and also the argument from the cosine and sine obtained as quotient of the real and imaginary components respectively divided by the modulus:

$$\cos\alpha = \frac{-\sqrt{2} + 2\sqrt{3}}{2\sqrt{5 + \sqrt{2} - \sqrt{6}}} \qquad \sin\alpha = \frac{\sqrt{2} + 2}{2\sqrt{5 + \sqrt{2} - \sqrt{6}}} \qquad \alpha = 1.0301...$$

$$z + t = 3.9823_{1.0301}$$

When multiplying complex numbers in binomial form, we apply the distributive property taken into account that the square of the imaginary unity is –1, e.g.:

$$(-2 + 5\, e_{12}\,)\,(\,3 - 4\, e_{12}\,) = -6 + 8\, e_{12} + 15\, e_{12} - 20\,(\, e_{12}\,)^2 = -6 + 20 + 23\, e_{12} = 14 + 23\, e_{12}$$

The exponential of an addition of arguments (real or complex) is equal to the product of exponential functions of each argument. Applying this characteristic property to the product of complex numbers in exponential form, we have:

$$z\,t = |z|\,\exp(\alpha\, e_{12}\,)\,|t|\,\exp(\,\beta\, e_{12}\,) = |z|\,|t|\,\exp[\,(\alpha + \beta)\, e_{12}\,]$$

from where the product of complex numbers in polar form is obtained by multiplying both moduli and adding both arguments.:

$$|z|_\alpha \; |t|_\beta = |z| \; |t|_{\alpha+\beta}$$

One may subtract 2π to the resulting argument in order to keep it between 0 and 2π. The product of two complex numbers z and t is the geometric operation consisting in the rotation of the parallelogram representing the first complex number until it touches the parallelogram representing the second complex number. When they contact in a unitary vector v (figure 3.3), the parallelogram formed by the other two vectors is the product of both complex numbers:

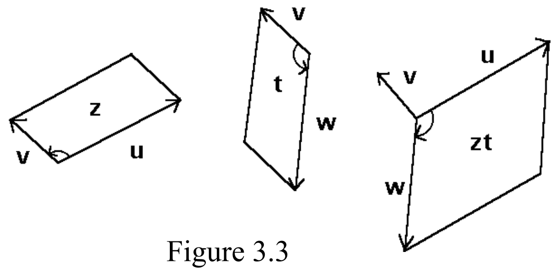


Figure 3.3

$$z = u\, v \qquad\qquad t = v\, w \qquad\qquad v^2 = 1$$

$$z\, t = u\, v\, v\, w = u\, w$$

This geometric construction is always possible because a parallelogram can be lengthened or widened maintaining the area so that one side has unity length.

The *conjugate* of a complex number (symbolised with an asterisk) is that number whose imaginary part has opposite sign:

$$z = a + b\, e_{12} \qquad\qquad z^* = a - b\, e_{12}$$

The geometric meaning of the conjugation is a permutation of the vectors whose product is the complex number (figure 3.4). In this case, the inner product is preserved while the outer product changes the sign. The product of a complex number and its conjugate is the square of the modulus:

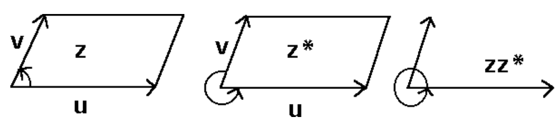


Figure 3.4

$$z\, z^* = u\, v\, v\, u = u^2 = |z|^2$$

The quotient of complex numbers is defined as the product by the inverse. The inverse of a complex number is equal to the conjugate divided by the square of the modulus:

$$z^{-1} = \frac{z^*}{|z|^2} = \frac{a - b\, e_{12}}{a^2 + b^2}$$

Then, let us see an example of quotient of complex numbers:

$$\frac{-2 + 5\, e_{12}}{3 - 4\, e_{12}} = \frac{(-2 + 5\, e_{12})(3 + 4\, e_{12})}{3^2 + (-4)^2} = \frac{-26 + 7\, e_{12}}{25}$$

With the polar form, the quotient is obtained by dividing moduli and subtracting arguments:

$$\frac{|z|_\alpha}{|t|_\beta} = \left(\frac{|z|}{|t|}\right)_{\alpha-\beta}$$

The best way to calculate the power of a complex number with natural exponent is through the polar form, although for low exponents the binomial form and the Newton formula is often used, e.g.:

$$(2 - 3\,e_{12})^3 = 2^3 - 3\cdot 2^2\,e_{12} + 3\cdot 2\,e_{12}{}^2 - e_{12}{}^3 = 8 - 12\,e_{12} - 6 + e_{12} = 2 - 11\,e_{12}$$

From the characteristic property of the exponential function it follows that the modulus of a power of a complex number is the power of its modulus, and the argument of this power is the argument of the complex number multiplied by the exponent:

$$\left(|z|_\alpha\right)^n = |z|^n{}_{n\alpha}$$

This is a very useful rule for large exponents, e.g.:

$$(2 + 2\,e_{12})^{1000} = (2\sqrt{2}_{\pi/4})^{1000} = [(2\sqrt{2}\,)^{1000}]_{250\,\pi} = 2^{1500}{}_0 = 2^{1500}$$

When the argument exceeds 2π, divide by this value and take the remainder, in order to have the argument within the period $0<\alpha<2\pi$.

Since a root is the inverse operation of a power, its value is obtained by extracting the root of the modulus and dividing the argument by the index n. But a complex number of argument α may be also represented by the arguments $\alpha+2\,\pi\,k$. Their division by the index n yields n different arguments within a period, corresponding to n different roots:

$$\sqrt[n]{z_\alpha} = \sqrt[n]{|z|}_{\;(\alpha+2\pi k)/n} \qquad k = 0, \ldots n-1$$

For example, the cubic roots of 8 are $\sqrt[3]{8_0} = \{2_0, 2_{2\pi/3}, 2_{4\pi/3}\}$. In the complex plane, the n-th roots of every complex number are located at the n vertices of a regular polygon.

Permutation of complex numbers and vectors

The permutative property of the vectors is intimately related with the commutative property of the product of complex numbers. Let z and t be complex numbers and a, b, c and d vectors fulfilling:

$$z = a\,b \qquad\qquad t = c\,d$$

Then the following equalities are equivalent:

$$z\,t = t\,z \qquad \Leftrightarrow \qquad a\,b\,c\,d = c\,d\,a\,b$$

A complex number z and a vector c do not commute, but they can be permuted by conjugating the complex number:

$$z\,c = a\,b\,c = c\,b\,a = c\,z^*$$

Every real number commute with any vector. However every imaginary number anticommute with any vector, because the imaginary unity e_{12} anticommute with e_1 as well as with e_2:

$$z\,c = -\,c\,z \qquad z \text{ imaginary}$$

The complex plane

In the *complex plane*, the complex numbers are represented taking the real component as the abscissa and the imaginary component as the ordinate. The vectorial plane differs from the complex plane in the fact that the vectorial plane is a plane of absolute directions whereas the complex plane is a plane of relative directions with respect to the real axis, to which we may assign any direction. As explained in more detail in the following chapter, the unitary complex numbers are rotation operators applied to vectors. The following equality shows the ambivalence of the Cartesian coordinates in the Euclidean plane:

$$e_1\left(x + y\,e_{12}\right) = x\,e_1 + y\,e_2$$

Due to a careless use, often the complex numbers have been improperly thought as vectors on the plane, furnishing the confusion between the complex and vector planes to our pupils. It will be argued that this has been very fruitful, but this argument cannot satisfy geometers, who search the fundamentals of the geometry. On the other hand, some physical magnitudes of a clearly vectorial kind have been taken improperly as complex numbers, specially in quantum mechanics. Because of this, I'm astonished when seeing how the inner and outer products transform in a special commutative and anti-commutative products of complex numbers. The relation between vectors and complex numbers is stated in the following way: If u is a fixed unitary vector, then every vector a is mapped to a unique complex z fulfilling:

$$a = u\,z \qquad \text{with} \quad u^2 = 1$$

Also other vector b is mapped to a complex number t:

$$b = u\,t$$

The outer and inner products of the vectors a and b can be written now using the complex numbers z and t:

$$a \wedge b = \frac{1}{2}(\,a\,b - b\,a\,) = \frac{1}{2}\;(\,u\,z\,u\,t - u\,t\,u\,z\,) = \frac{1}{2}u^2\,(\,z^*\,t - t^*\,z\,) =$$

$$= \frac{1}{2}\ (z^*\, t - t^*\, z) = (z_R\, t_I - z_I\, t_R)\, e_{12}$$

$$a \cdot b = \frac{1}{2} (a\, b + b\, a) = \frac{1}{2} (u\, z\, u\, t + u\, z\, u\, t) = \frac{1}{2}\ (z^*\, t + t^*\, z) = z_R\, t_R + z_I\, t_I$$

where z_R, t_R, z_I, t_I are the real and imaginary components of z and t. These products have been called improperly scalar and exterior products of complexes. So, I repeat again that complex quantities must be distinguished from vectorial quantities, and relative directions (complex numbers) from absolute directions (vectors). A guide for doing this is the reversion, under which the vectors are reversed while the complex number are not[1].

Complex analytic functions

The complex numbers are a commutative algebra where we can study functions as for the real numbers. A function $f(x)$ is said to be analytical if its complex derivative exists:

$$f(\mathrm{z}) \text{ analytical at } z_0 \quad \Leftrightarrow \quad \exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

This means that the derivative measured in any direction must give the same result. If $f(x) = a + b\, e_{12}$ and $z = x + y\, e_{12}$, the derivatives following the abscissa and ordinate directions must be equal:

$$f'(z) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial x} e_{12} = -\frac{\partial a}{\partial y} e_{12} + \frac{\partial b}{\partial y}$$

whence the Cauchy-Riemann conditions are obtained:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \text{and} \qquad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$$

Due to its linearity, now the derivative in any direction are also equal. A consequence of these conditions is the fact that the sum of both second derivatives (the Laplacian) vanishes, that is, both components are *harmonic* functions:

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} = 0$$

[1] A physical example is the alternating current. The voltage V and intensity I in an electric circuit are continuously rotating vectors. The energy E dissipated by the circuit is the inner product of both vectors, $E = V \cdot I$. The impedance Z of the circuit is of course a complex number (it is invariant under a reversion). The intensity vector can be calculated as the geometric product of the voltage vector multiplied by the inverse of the impedance $I = V\, Z^{-1}$. If we take as reference a continuously rotating direction, then V and I are replaced by pseudo complex numbers, but properly they are vectors.

The values of a harmonic function (therefore the value of $f(z)$) within a region are determined by those values at the boundary of this region. We will return to this matter later. The typical example of analytic function is the complex exponential:

$$\exp(x + y\, e_{12}) = \exp(x)(\cos y + e_{12} \sin y)$$

which is analytic in all the plane. The logarithm function is defined as the inverse function of the exponential. Since $z = |z| \exp(e_{12}\, \varphi)$ where φ is the argument of the complex, the principal branch of the logarithm is defined as:

$$\log z = \log |z| + e_{12} \varphi \qquad 0 \le \varphi < 2\pi$$

Also $\varphi + 2\pi k$ (k integer) are valid arguments for z yielding another branches of the logarithm[2]. In Cartesian coordinates:

$$\log(x + y\, e_{12}) = \log \sqrt{x^2 + y^2} + e_{12} \arccos \frac{x}{\sqrt{x^2 + y^2}} \qquad 0 \le \varphi < \pi$$

$$\log(x + y\, e_{12}) = \log \sqrt{x^2 + y^2} + e_{12} \left(\arccos \frac{x}{\sqrt{x^2 + y^2}} + \pi \right) \qquad \pi \le \varphi < 2\pi$$

At the positive real half axis, this logarithm is not analytic because it is not continuous.

Now let us see the Cauchy's theorem: if a function is analytic in a simply connected domain on the complex plane, then its integral following a closed way C within this domain is zero. If the analytic function is $f(z) = a + b\, e_{12}$ then the integral is:

$$\oint_C f(z) dz = \oint_C (a + b\, e_{12})(dx + dy\, e_{12}) = \oint_C (a\, dx - b\, dy) + e_{12} \oint_C (a\, dy + b\, dx)$$

Since C is a closed way, we may apply the Green theorem to write:

$$= -\iint_D \left(\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y} \right) dx\, dy + e_{12} \iint_D \left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) dx\, dy = 0$$

where D is the region bounded by the closed way C. Since $f(z)$ fulfils the analyticity conditions everywhere within D, the integral vanishes.

From here the following theorem is deduced: if $f(z)$ is an analytic function in a simply connected domain D and z_1 and z_2 are two points of D, then the definite integral between these points has a unique value independently of the integration trajectory, which is equal to the difference of the values of the primitive $F(z)$ at both points:

$$\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1) \qquad \text{if} \qquad f(z) = \frac{dF(z)}{dz}$$

---

[2] The logarithm is said to be *multi-valued*. This is also the case of the roots $\sqrt[n]{z}$.

If $f(z)$ is an analytic function (with a unique value) inside the region D bounded by the closed path C, then the *Cauchy integral formula* is fulfilled for a counterclockwise path orientation:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0)$$

Obviously, the integral does not vanish because the integrand is not analytic at z_0 . However, it is analytic at the other points of the region D, so that the integration path from its beginning to its end passes always through an analyticity region, and by the former theorem the definite integral must have a constant value, independently of the fact that both extremes coincide. Now we integrate following a circular path $z = z_0 + r\exp(e_{12}\varphi)$, where the radius r is a real constant and the angle φ is a real variable. The evaluation of the integral gives $f(z_0)$:

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(e_{12}\varphi))\, d\varphi = f(z_0)$$

because we can take any radius and also the limit $r \to 0$. The consequence of this theorem is immediate: the values of $f(z)$ at a closed path C determine its value at any z_0 inside the region bounded by C. This is a characteristic property of the harmonic functions, already commented above.

Let us rewrite the Cauchy integral formula in a more suitable form:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t - z}\, dt = f(z)$$

The first and successive derivative with respect to z are:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{(t - z)^2}\, dt = f'(z) \qquad \frac{n!}{2\pi\, e_{12}} \oint_C \frac{f(t)}{(t - z)^{n+1}}\, dt = f^{n}(z)$$

For $z = 0$ we have:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t}\, dt = f(0) \qquad \frac{n!}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t^{n+1}}\, dt = f^{n}(0)$$

Now we see that these integrals always exist if $f(z)$ is analytic, that is, all the derivatives exist at the points where the function is analytic. In other words, the existence of the first derivative (analyticity) implies the existence of those with higher order.

The Cauchy integral formula may be converted into a power series of z:

$$f(z) = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t - z}\, dt = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)\, dt}{t\,(1 - z/t)} = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t} \sum_{k=0}^{\infty} \left(\frac{z}{t}\right)^k dt$$

Rewriting this expression we find the Taylor series:

$$f(z)=\frac{1}{2\pi\, e_{12}}\sum_{k=0}^{\infty} z^k \oint_C \frac{f(t)}{t^{k+1}}\,dt=\sum_{k=0}^{\infty}\frac{f^k(0)}{k\,!}\,z^k$$

The only assumption made in the deduction is the analyticity of $f(z)$. So this series is convergent within the largest circle centred at the origin where the function is analytic (that is, the convergence circle touches the closest singular point). The Taylor series is unique for any analytic function. On the other hand, every analytic function has a Taylor series.

Instead of the origin we can take a series centred at another point z_0. In this case, following the same way as above, one arrives to the MacLaurin series:

$$f(z)=\sum_{k=0}^{\infty}\frac{f^k(z_0)}{k\,!}(z-z_0)^k$$

For instance, the Taylor series ($z_0 = 0$) of the exponential, taking into account the fact that all the derivatives are equal to the exponential and exp(0) = 1, is:

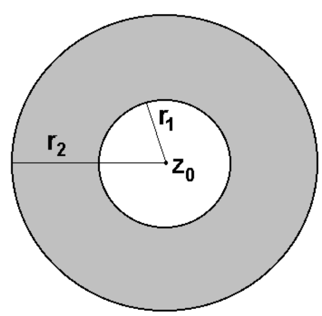
$$\exp(z)=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$$

The exponential has not any singular point. Then the radius of convergence is infinite.

In order to find a convergent series for a function which is analytic in an annulus although not at its centre (for $r_1 < |z - z_0| < r_2$ as shown in figure 3.5), we must add powers with negative exponents, obtaining the Lauren series:

$$f(z)=\sum_{k=-\infty}^{\infty} a_k\,(z-z_0)^k$$

Figure 3.5



The Lauren series is unique, and coincident with the McLaurin series if the function has not any singularity at the central region. The coefficients are obtained in the same way as above:

$$a_k=\frac{1}{2\pi\, e_{12}}\oint_C \frac{f(t)\,dt}{t^{k+1}}$$

where the path C encloses the central circle. When $f(x)$ is not analytic at some point of this central circle (e.g. z_0), the powers of negative exponent appear in the Laurent series. The coefficient a_{-1} is called the *residue* of f. The Lauren series is the addition of a series of powers with negative exponent and the McLaurin series:

$$f(z)=\sum_{k=-\infty}^{\infty} a_k\,(z-z_0)^k=\sum_{k=-\infty}^{-1} a_k\,(z-z_0)^k+\sum_{k=0}^{\infty} a_k\,(z-z_0)^k$$

and is convergent only when both series are convergent, so that the annulus goes from the radius of convergence of the first series (r_1) to the radius of convergence of the McLaurin series (r_2).

Let us review singular points, the points where an analytic function is not defined. An isolated singular point may be a *removable singularity*, an *essential singularity* or a *pole*. A point z_0 is a removable singularity if the limit of the function at this point exists and, therefore, we may remove the singularity taking the limit as the value of the function at z_0. A point z_0 is a pole of a function $f(z)$ if it is a zero of the function $1/f(z)$. Finally, z_0 is an essential singularity if both limits of $f(z)$ and $1/f(z)$ at z_0 do not exist.

The Lauren series centred at a removable singularity has not powers with negative exponent. The series centred at a pole has a finite number of powers with negative exponent, and that centred at an essential singularity has an infinite number of powers with negative exponent. Let us see some examples. The function $\sin z / z$ has a removable singularity at $z = 0$:

$$\frac{\sin z}{\mathrm{z}} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \ldots$$

So the series has only positive exponents.

The function $\exp(1/z)$ has an essential singularity at $z = 0$. From the series of $\exp(z)$, we obtain a Lauren series with an infinite number of powers with negative exponent by changing z for $1/z$. Also the radius of convergence is infinite:

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots$$

Finally, the function $1/z^2(z-1)$ has a pole at $z = 0$. Its Lauren series:

$$\frac{1}{z^2(z-1)} = -\frac{1}{z^2}\left(1 + z + z^2 + z^3 + z^4 \ldots\right) = -\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 \ldots$$

is convergent for $0<|z|<1$ since the function has another pole at $z = 1$.

To see the importance of the residue, let us calculate the integral of a function through an annular way from its Lauren series:

$$\oint_C f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_0)^k\,dz$$

For $k\geq 0$ the integral is zero because $(z - z_0)^k$ is analytic in the whole domain enclosed by C. For $k<-2$ the integral is also zero because the path is inside a region where the powers are analytic:

$$\oint_C (z - z_0)^k\,dz = \begin{matrix} lim \\ z_1 \to z_2 \end{matrix} \left[\frac{(z - z_0)^{k-1}}{k-1}\right]_{z_1}^{z_2} = 0$$

However $k = -1$ is a special case. Taking the circular path $z = r\exp(e_{12}\varphi)$ ($r_1 < r < r_2$) we have:

$$\oint_C \frac{dz}{z - z_0} = e_{12} \int_0^{2\pi} d\varphi = 2\pi\ e_{12}$$

So that the *residue theorem* is obtained:

$$\oint_C f(z)dz = 2\pi\ e_{12}\ a_{-1}$$

where a_{-1} is the coefficient of the Lauren series centred at the pole. If the path C encloses some poles, then the integral is proportional to the sum of the residues:

$$\oint_C f(z)dz = 2\pi\ e_{12} \sum \text{residues}$$

Let us see the case of the last example. If C is a path enclosing $z = 0$ and $z = 1$ then the residue for the first pole is 1 and that for the second pole –1 (for a counterclockwise path) so that the integral vanishes:

$$\oint_C \frac{dz}{z^2(z-1)} = \oint_C \left(\frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2} \right) dz = \oint_C \frac{dz}{z-1} - \oint_C \frac{dz}{z} = 0$$

The fundamental theorem of algebra

Firstly let us prove the *Liouville's theorem*: if $f(x)$ is analytic and bounded in the whole complex plane then it is a constant. If $f(x)$ is bounded we have:

$$|f(x)| < M$$

The derivative of $f(x)$ is always given by:

$$f'(z) = \frac{1}{2\pi\ e_{12}} \oint_C \frac{f(t)}{(t-z)^2}\, dt$$

Following the circular path $t - z = r\exp(e_{12}\varphi)$ we have:

$$f'(z) = \frac{1}{2\pi\ r} \int_0^{2\pi} f(r\exp(e_{12}\varphi))\ \exp(-e_{12}\varphi)\, d\varphi$$

Using the inequality $\left| \int f(z)\, dz \right| \leq \int |f(z)|\, dz$, we find:

$$|f'(z)| \leq \frac{1}{2\pi\ r} \int_0^{2\pi} |f(r\exp(e_{12}\varphi))|\ d\varphi \leq \frac{1}{2\pi\ r} \int_0^{2\pi} M\ d\varphi = \frac{M}{2\pi\ r}$$

Since the function is analytic in the entire plane, we may take the radius r as large as we wish. In consequence, the derivative must be null and the function constant, which is the proof of the theorem.

A main consequence of the Liouville's theorem is the *fundamental theorem of algebra*: any polynomial of degree n has always n zeros (not necessarily different):

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0 \quad \Rightarrow \quad \exists\ z_i \qquad i \in \{1, .., n\} \qquad p(z_i) = 0$$

where $a_0, a_1, a_2, \ldots a_n$ are complex coefficients. For real coefficients, the zeros are whether real or pairs of conjugate complex numbers. The proof is by supposing that $p(z)$ has not any zero. In this case $f(z) = 1/p(z)$ is analytic and bounded (because $p(z) \rightarrow 0$ for $|z| \rightarrow \infty$) in the whole plane. From the Liouville's theorem $f(z)$ and $p(z)$ should be constant becoming in contradiction with the fact that $p(z)$ is a polynomial. In conclusion $p(z)$ has at least one zero.

According to the division algorithm, the division of the polynomial $p(z)$ by $z - b$ decreases the degree of the quotient $q(z)$ by a unity, and yields a complex number r as remainder:

$$p(z) = (z - b)\, q(z) + r$$

The substitution of z by b gives:

$$p(b) = r$$

That is, the remainder of the division of a polynomial by $z - b$ is equal to its numerical value for $z = b$. On the other hand, if b is a zero z_i, the remainder vanishes and we have an exact division:

$$p(z) = (z - z_i)\, q(z)$$

Again $q(z)$ has at least one zero. In each division, we find a new zero and a new factor, so that the polynomial completely factorises with as many zeros and factors as the degree of the polynomial, which ends the proof:

$$p(z) = a_n \prod_{i=1}^{n} (z - z_i) \qquad n = \text{degree of } p(z)$$

For example, let us calculate the zeros of the polynomial $z^3 - 5\, z^2 + 8\, z - 6$. By the Ruffini method we find the zero $z = 3$:

$$\begin{array}{c|cccc}
 & 1 & -5 & 8 & -6 \\
3 & & 3 & -6 & 6 \\
\hline
 & 1 & -2 & 2 & \multicolumn{1}{|c}{0}
\end{array}$$

The zeros of the quotient polynomial $z^2 - 2z - 2$ are obtained through the formula of the equation of second degree:

$$z = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = 1 \pm \sqrt{-1} = 1 \pm e_{12}$$

yielding a pair of conjugate zeros. Then the factorisation of the polynomial is:

$$z^3 - 5\,z^2 + 8\,z - 6 = \;(\,z - 3\,)\,(\,z - [1 + e_{12}\,]\,)\,(\,z - [1 - e_{12}\,]\,)$$

Exercises

3.1 Multiply the complex numbers $z = 1 + 3\,e_{12}$ and $t = -2 + 2\,e_{12}$. Draw the geometric figure of their product and check the result found.

3.2 Prove that the modulus of a complex number is the square root of the product of this number by its conjugate.

3.3 Solve the equation $x^4 - 1 = 0$. Being of fourth degree, you must obtain four complex solutions.

3.4 For which natural values of n is fulfilled the following equation?

$$(\,1 + e_{12}\,)^n + (\,1 - e_{12}\,)^n = 0$$

3.5 Find the cubic roots of $-3 + 3\,e_{12}$.

3.6 Solve the equation:

$$z^2\; + (\,-3 + 2\,e_{12}\,)\,z + 5 - e_{12} = 0$$

3.7 Find the analytical extension of the real functions sin x and cos x.

3.8 Find the Taylor series of $\log(1 + \exp(-z))$.

3.9 Calculate the Lauren series of $\dfrac{1}{z^2 + 2z - 8}$ in the annulus $1 < |\,z - 2\;| < 4$.

3.10 Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n\,(z+1)^n}$ and its analytic function.

3.11Calculate the Lauren series of $\dfrac{\sin z}{z^2}$ and the annulus of convergence.

3.12 Prove that if $f(z)$ is analytic and does not vanish then it is a conformal mapping.

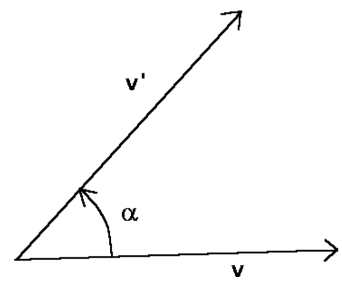
4. TRANSFORMATIONS OF VECTORS

The transformations of vectors are mappings from the vector plane to itself. Those transformations preserving the modulus of vectors, such as rotations and reflections, are called *isometries* and those which preserve angles between vectors are said to be *conformal*. Besides rotations and reflections, the inversions and dilatations are also conformal transformations.

Rotations

A *rotation* through an angle α is the geometric operation consisting in turning a vector until it forms an angle α with the previous orientation. The positive direction of angles is counterclockwise (figure 4.1). Under rotations the modulus of any vector is preserved. According to the definition of geometric product, the multiplication of a vector v by a unitary complex number with argument α produces a vector v' rotated through an angle α with regard to v.

Figure 4.1



$$v' = v\, 1_\alpha = v\,(\cos\alpha + e_{12}\sin\alpha)$$

This algebraic expression for rotations, when applied to a real or complex number instead of vector, modifies its value. However, real numbers are invariant under rotations and the parallelograms can be turned without changing the complex number which they represent. Therefore this expression for rotations, although being useful for vectors, is not valid for complex numbers. In order to remodel it, we factorise the unitary complex number into a product of two complex numbers with half argument. According to the permutative property, we can permute the vector and the first complex number whenever writing the conjugate:

$$v' = v\, 1_\alpha = v\, 1_{\alpha/2}\, 1_{\alpha/2} = 1_{-\alpha/2}\ v\, 1_{\alpha/2} = \left(\cos\frac{\alpha}{2} - e_{12}\sin\frac{\alpha}{2}\right) v \left(\cos\frac{\alpha}{2} + e_{12}\sin\frac{\alpha}{2}\right)$$

The algebraic expression for rotations now found preserves complex numbers:

$$z' = 1_{-\alpha/2}\ z\, 1_{\alpha/2} = z$$

Let us calculate the rotation of the vector $4\,e_1$ through $2\pi / 3$ by multiplying it by the unitary complex with this argument:

$$v' = 4\,e_1\left(\cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}\right) = 4\,e_1\left(-\frac{1}{2} + e_{12}\frac{\sqrt{3}}{2}\right) = -2\,e_1 + 2\sqrt{3}\,e_2$$

On the other hand, using the half angle $\pi/3$ we have:

$$v' = \left(\cos\frac{\pi}{3} - e_{12}\sin\frac{\pi}{3}\right) 4\, e_1 \left(\cos\frac{\pi}{3} + e_{12}\sin\frac{\pi}{3}\right)$$

$$= \left(\frac{1}{2} - e_{12}\frac{\sqrt{3}}{2}\right) 4\, e_1 \left(\frac{1}{2} + e_{12}\frac{\sqrt{3}}{2}\right) = -2\, e_1 + 2\sqrt{3}\, e_2$$

With the expression of half angle, it is not necessary that the complex number has unity modulus because:

$$z = |\,z\,|\, 1_{\alpha/2} \qquad\qquad z^{-1} = 1_{-\alpha/2}\ |\,z\,|^{-1}$$

Then, the rotation through an angle α can be written as:

$$v' = z^{-1}\, v\, z$$

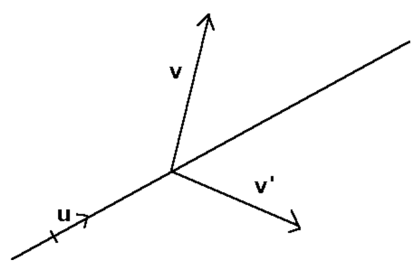
The composition of two successive rotations implies the product of both complex numbers, whose argument is the addition of the angles of both rotations.

$$v'' = 1_{-\beta/2}\, v'\, 1_{\beta/2} = 1_{-\beta/2}\, 1_{-\alpha/2}\, v\, 1_{\alpha/2}\, 1_{\beta/2} = 1_{-(\alpha+\beta)/2}\, v\, 1_{(\alpha+\beta)/2}$$

Reflections

A *reflection* of a vector in a direction is the geometric transformation which preserves the component having this direction and changes the sign of the perpendicular component (figure 4.2). The product of proportional vectors is commutative and that of orthogonal vectors is anti-commutative. Because of this, the reflected vector v' may be obtained as the multiplication of the vector v by the unitary vector u of the reflection axis at the left and right hand sides:

Figure 4.2



$$v' = u\, v\, u = u\,(\,v_{\parallel} + v_{\perp}\,)\, u = u\, v_{\parallel}\, u + u\, v_{\perp}\, u = v_{\parallel} - v_{\perp} \qquad \text{with} \quad u^2 = 1$$

where $v_{\parallel}$ and $v_{\perp}$ are the components of v proportional and perpendicular to u respectively.

Instead of the unitary vector u, any vector d having the axis direction can be introduced in the expression for reflections whenever we write its inverse at the left side of the vector:

$$v' = \frac{d\, v\, d}{|\,d\,|^2} = d^{-1} v\, d$$

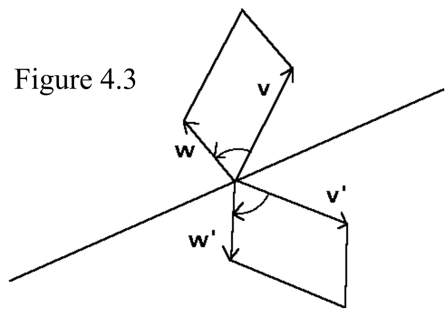
Although the reflection does not change the absolute value of the angle between vectors, it changes its sign. Under reflections, real numbers remain invariant but the complex numbers become conjugate because a reflection generates a symmetric parallelogram (figure 4.3) and changes the sign of the imaginary part:

$z = a + b\, e_{12}$ $\qquad$ *a, b* real

$$z' = \frac{d\,(a + b\,e_{12})\,d}{|\,d\,|^2}$$

$$= \frac{d^2 a - d^2 b\, e_{12}}{|\,d\,|^2} = a - b\, e_{12} = z^*$$

Figure 4.3



For example, let us calculate the reflection of the vector 3 e_1 + 2 e_2 with regard to the direction $e_1 - e_2$. The resulting vector will be:

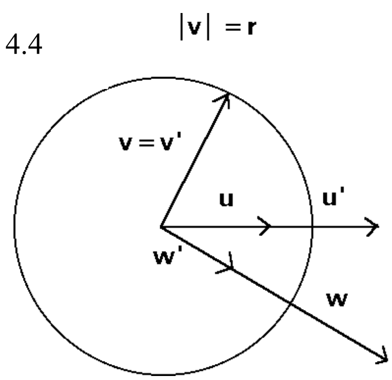
$$v' = \frac{1}{2}\,(\, e_1 - e_2\,)\,(\, 3\, e_1 + 2\, e_2\,)\,(\, e_1 - e_2\,) = \frac{1}{2}\,(\, e_1 - e_2\,)\,(\, 1 - 5\, e_{12}\,) = -\,2\, e_1 - 3\, e_2$$

Inversions

The *inversion* of radius *r* is the geometric transformation which maps every vector *v* on $r^2\, v^{-1}$, that is, on a vector having the same direction but with a modulus equal to r^2 / $|\,v\,|$:

$v' = r^2\, v^{-1}$ $\qquad$ *r* real

Figure 4.4



This operation is a generalisation of the inverse of a vector in the geometric algebra (radius $r = 1$). It is called inversion of radius *r*, because all the vectors with modulus *r*, whose heads lie on a circumference with this radius, remain invariant (figure 4.4). The vectors whose heads are placed inside the circle of radius *r* transform into vectors having the head outside and reciprocally.

The inversion transforms complex numbers into proportional complex numbers with the same argument (figure 4.5):

$v' = r^2\, v^{-1}$ $\qquad$ $w' = r^2\, w^{-1}$ $\qquad$ $z = v\, w$

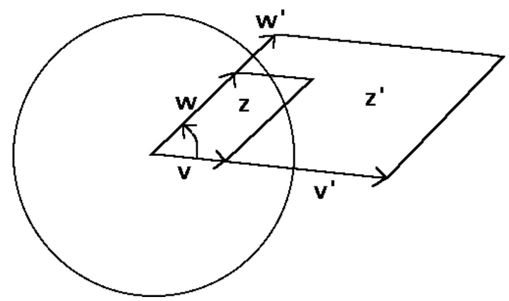
$$z' = v'\, w' = r^4\, v^{-1}\, w^{-1} = r^4\, v\, w\, v^{-2}\, w^{-2} = \frac{r^4\, z}{|\,z\,|^2}$$

Dilatations

The *dilatation* is that geometric transformation which enlarges or shortens a vector, that is, it increases (or decreases) k times the modulus of any vector preserving its orientation. The dilatation is simply the product by a real number k

$$v' = k\,v \qquad k \text{ real}$$

Figure 4.5



The most transformations of vectors that will be used in this book are combinations of these four elementary transformations. Many physical laws are invariant under some of these transformations. In geometry, from the vector transformations we define the transformations of points on the plane, indispensable for solving geometric problems.

Exercises

4.1 Calculate with geometric algebra what is the composition of a reflection with a rotation.

4.2 Prove that the composite of two reflections in different directions is a rotation.

4.3 Consider the transformation in which every vector v multiplied by its transformed v' is equal to a constant complex z^2. Resolve it into elementary transformations.

4.4 Apply a rotation of $2\pi/3$ to the vector $-3\, e_1 + 2\, e_2$ and find the resulting vector.

4.5 Find the reflection of the former vector in the direction $e_1 + e_2$.

SECOND PART: THE GEOMETRY OF THE EUCLIDEAN PLANE

A complete description of the Euclidean plane needs not only directions (given by vectors) but also positions. Points represent locations on the plane. Then, the plane R^2 is the set of all the points we may draw on a fictitious sheet of infinite extension.

Recall that points are noted with capital Latin letters, vectors and complex numbers with lowercase Latin letters, and angles with Greek letters. A vector going from the point P to the point Q will be symbolised by PQ and the line passing through both points will be distinguished as $\overline{PQ}$.

Translations

A *translation* v is the geometric operation which moves a point P in the direction and length of the vector v to give the point Q:

$$+ : R^2 \times V_2 \rightarrow R^2$$

$$(P, v) \rightarrow Q = P + v$$

Then the vector v is the oriented segment going from P to Q. The set of points together with vectors corresponding to translations is called *the affine plane*[1] (R^2 , V_2, +). From this equality a vector is defined as a subtraction of two points, usually noted as PQ :

$$v = Q - P = PQ$$

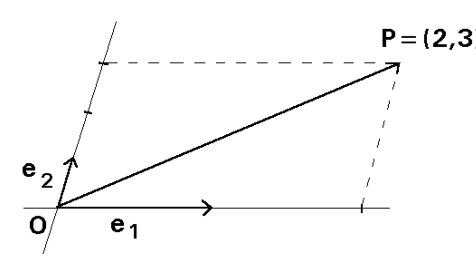
The sign of addition is suitable for translations, because the composition of translations results in an addition of their vectors:

$$Q = P + v \qquad R = Q + w \qquad R = (P + v) + w = P + (v + w)$$

5. POINTS AND STRAIGHT LINES

Coordinate systems

Figure 5.1



Given an origin of coordinates O, every point P on the plane is described by the *position vector* OP traced from the origin. The *coordinates* (c_1, c_2) of a point are the components of the postion vector in the chosen vector base $\{e_1, e_2\}$:

$$OP = c_1\, e_1 + c_2\, e_2$$

Then every point is given by means of a pair of

[1] The affine plane does not imply a priori an Euclidean or pseudo-Euclidean character.

coordinates:

$$P = (c_1, c_2)$$

For example, the position vector shown in the figure 5.1 is:

$$OP = 2\, e_1 + 3\, e_2$$

and hence the coordinates of P are (2, 3).

A *coordinate system* is the set $\{O;\, e_1,\, e_2\}$, that is, the origin of coordinates O together with the base of vectors. The coordinates of a point depend on the coordinate system to which they belong. If the origin or any base vector is changed, the coordinates of a point are also modified. For example, let us calculate the coordinates of the points P and Q in the figure 5.2. Both coordinate systems have the same vector base but the origin is different:

$$S = \{O;\, e_1,\, e_2\} \qquad S' = \{O';\, e_1,\, e_2\}$$

Since $OP = 2\, e_1 + 3\, e_2$ and $O'P = 2\, e_2$, the coordinates of the point P in the coordinate systems S and S' are respectively:

$$P = (2, 3)_S = (0, 2)_{S'}$$

Analogously:

$$OQ = -\,4\, e_1 + e_2 \quad \text{and} \quad O'Q = -\,6\, e_1$$

$$Q = (-4, 1)_S = (-6, 0)_{S'}$$

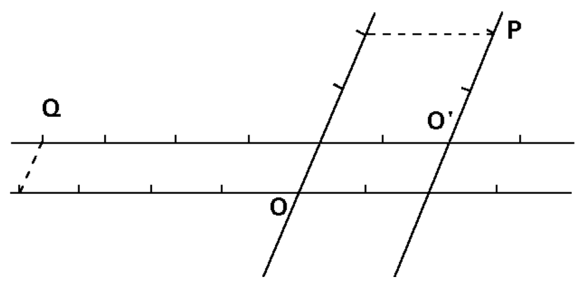


Figure 5.2

The *coordinate axes* are the straight lines passing through the origin and having the direction of the base vectors. All the points lying on a coordinate axis have the other coordinate equal to zero.

If a point Q is obtained from a point P by the translation v then:

$$Q = P + v \quad \Leftrightarrow \quad OQ = OP + v$$

that is, we must add the components of the translation vector v to the coordinates of the point P in order to obtain the coordinates of the point Q:

$$(q_1, q_2) = (p_1, p_2) + (v_1, v_2)$$

For example, let us apply the translation $v = 3\, e_1 - 5\, e_2$ to the point $P = (-6, 7)$:

$$Q = P + v = (-6, 7) + (3, -5) = (-3, 2)$$

On the other hand, given two points, the translation vector having these points as extremes is found by subtracting their coordinates. For instance, the vector from $P = (2, 5)$

to $Q = (-3, 4)$ is:

$$PQ = Q - P = (-3, 4) - (2, 5) = (-5, -1) = -5\, e_1 - e_2$$

Remember the general rule for operations between points and vectors:

coordinates = coordinates + components

components = coordinates – coordinates

Barycentric coordinates

Why is the plane described by complicated concepts such as translations and the coordinate system instead of using points as fundamental elements? This question was studied and answered by Möbius in *Der Barycentrische Calculus* (1827) and Grassmann in *Die Ausdehnungslehre* (1844). From the figure 5.3 it follows that:

$$OR = c_1\, e_1 + c_2\, e_2 = c_1\, OP + c_2\, OQ$$

When writing all vectors as difference of points and isolating the generic point *R*, we obtain:

$$R = (1 - c_1 - c_2)\, O + c_1\, P + c_2\, Q$$

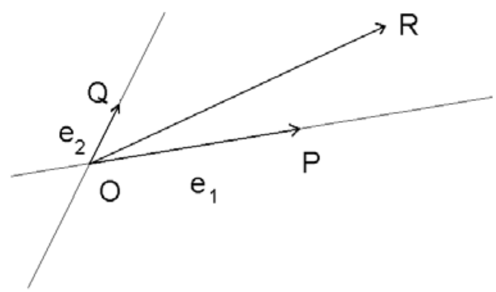


Figure 5.3

That is, a coordinate system is a set of three non-aligned points $\{O, P, Q\}$ so that every point on the plane is a linear combination of these three points with coefficients whose addition is equal to the unity. The coefficients $(1 - c_1 - c_2, c_1, c_2)$ are usually called *barycentric* coordinates, although they only differ from the usual coordinates in a third dependent coordinate.

Distance between two points and area

The *distance* between two points is the modulus of the vector going from one point to the other:

$$d(P, Q) = |PQ| = |Q - P|$$

The distance has the following properties:

1) The distance from a point to itself is zero: $d(P, P) = |PP| = 0$ and the reverse assertion: if the distance between two points is null then both points are coincident:

$$d(P, Q) = 0 \quad \Rightarrow \quad |PQ| = 0 \quad \Rightarrow \quad Q - P = 0 \quad \Rightarrow \quad P = Q$$

2) The distance has the symmetrical property:

$$d(P, Q) = |PQ| = |QP| = d(Q, P)$$

3) The distance fulfils the *triangular inequality*: the addition of the lengths of any two sides of a triangle is always higher than or equal to the length of the third side (figure 5.4).

$$d(P, Q) + d(Q, R) \geq d(P, R)$$

that is:

$$|PQ| + |QR| \geq |PR| = |PQ + QR|$$

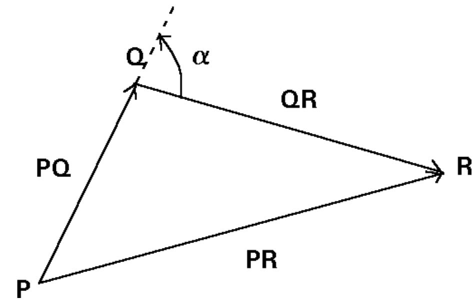


Figure 5.4

The proof of the triangular inequality is based on the fact that the product of the modulus of two vectors is always higher than or equal to the inner product:

$$|PQ|\ |QR| \geq PQ \cdot QR = |PQ|\ |QR| \cos\alpha$$

Adding the square of both vectors to the left and right hand sides, one has:

$$PQ^2 + QR^2 + 2\ |PQ|\ |QR| \geq PQ^2 + QR^2 + 2\ PQ \cdot QR$$

$$(|PQ| + |QR|)^2 \geq (PQ + QR)^2$$

$$|PQ| + |QR| \geq |PQ + QR|$$

For any kind of coordinate system the distance is calculated through the inner product:

$$d^2(P, Q) = PQ^2 = PQ \cdot PQ$$

The oriented area of a parallelogram is the outer product of both non parallel sides. If P, Q and R are three consecutive vertices of the parallelogram, the area is:

$$A = PQ \wedge QR$$

Then, the area of the triangle with these vertices is the half of the parallelogram area.

$$A = \frac{1}{2} PQ \wedge QR$$

If the coordinates are given in the system formed by three non-linear points $\{O, X, Y\}$:

$$P = (\ 1 - x_P - y_P\ )\ O + x_P\ X + y_P\ Y$$

$$Q = (\ 1 - x_Q - y_Q\ )\ O + x_Q\ X + y_Q\ Y$$

$$R = (1 - x_R - y_R)\, O + x_R\, X + y_R\, Y$$

it is easy to prove that:

$$PQ \wedge QR = \begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix} OX \wedge OY$$

Then the absolute value of the area of the triangle *PQR* is equal to the product of the absolute value of the determinant of the coordinates and the area of the triangle formed by the three base points:

$$|\,A\,| = \left|\,\begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix}\,\right| \frac{|OX \wedge OY|}{2}$$

Condition of alignment of three points

Three points *P*, *Q*, *R* are said to be *aligned*, that is, they lie on a line, if the vectors *PQ* and *PR* are proportional, and therefore commute.

$$P, Q, R \text{ aligned} \quad \Leftrightarrow \quad PR = k\, PQ \quad \Leftrightarrow \quad PQ \wedge PR = 0 \quad \Leftrightarrow$$

$$PQ\, PR - PR\, PQ = 0 \quad \Leftrightarrow \quad PQ\, PR = PR\, PQ \quad \Leftrightarrow \quad PR = PQ^{-1}\, PR\, PQ$$

The first equation yields a proportionality between components of vectors:

$$\frac{y_Q - y_P}{x_Q - x_P} = \frac{y_R - y_P}{x_R - x_P}$$

The second equation means that the area of the triangle *PQR* must be zero:

$$\begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix} = 0$$

that is, the barycentric coordinates of the three points are linearly dependent. According to the previous chapter, the last equality means that the vector *PR* remains unchanged after a reflection in the direction *PQ*. Obviously, this is only feasible when *PR* is proportional to *PQ*, that is, when the three points are aligned.

Cartesian coordinates

The coordinates (x, y) of any system having an orthonormal base of vectors are called *Cartesian coordinates* from Descartes[2]. The horizontal coordinate x is called *abscissa* and the vertical coordinate y *ordinate*. In this kind of bases, the square of a vector is equal to the sum of the squares of both components, leading to the following formula for the distance between two points:

$$P = (x_P, y_P) \qquad Q = (x_Q, y_Q)$$

$$d(P,Q) = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$

For example, the triangle with vertices $P = (2, 5)$, $Q = (3, 2)$ and $R = (-4, 1)$ have the following sides:

$$PQ = Q - P = (3, 2) - (2, 5) = (1, -3) = e_1 - 3\, e_2 \qquad |PQ| = \sqrt{10}$$

$$QR = R - Q = (-4, 1) - (3, 2) = (-7, -1) = -7\, e_1 - e_2 \qquad |QR| = \sqrt{50}$$

$$RP = P - R = (2, 5) - (-4, 1) = (6, 4) = 6\, e_1 + 4\, e_2 \qquad |PR| = \sqrt{40}$$

The area of the triangle is the half of the outer product of any pair of sides:

$$A = \frac{1}{2} PQ \wedge QR = \frac{1}{2}\,(e_1 - 3\, e_2) \wedge (-7\, e_1 - e_2) = -11\, e_{12} \qquad |A| = 11$$

Vectorial and parametric equations of a line

The condition of alignment of three points is the starting point to deduce the equations of a straight line. A line is determined whether by two points or by a point and a direction defined by its *direction vector*, which is not unique because any other proportional vector will also be a direction vector for this line. Known the direction vector v and a point P on the line, the *vectorial equation* gives the generic point R on the line:

$$r\!: \{v, P\} \qquad P = (x_p, y_P) \qquad R = (x, y)$$

$$PR = k\, v$$

Separating coordinates, the *parametric equation* of the line is obtained:

$$(x, y) = (x_p, y_p) + k\,(v_1, v_2)$$

Here the position of a point on the line depends on the real parameter k, usually identified

[2] If the x-axis is horizontal and the y-axis vertical, an orthonormal base of vectors is called *canonical*.

with the time in physics.

Knowing two points P, Q on the line, a direction vector v can be obtained by subtraction of both points:

$$r: \{P, Q\} \qquad v = Q - P$$

In this case, the vectorial and parametric equations become:

$$PR = k\, PQ \quad \Leftrightarrow \quad R = P + k\,(Q - P) = P\,(1 - k) + k\,Q$$

The parameter k indicates in which proportion the point R is closer to P than to Q. For example, the midpoint of the segment PQ can be calculated with $k = 1/2$. The figure 5.5 shows where the point R is located on the line PQ as a function of the parameter k. In this way, all the points of the line are mapped to the real numbers incorporating the order relation and the topologic properties of this set to the line[3].

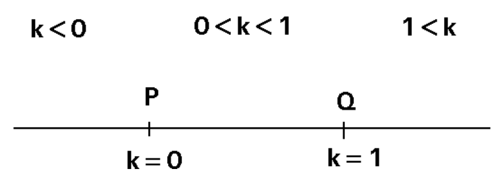


Figure 5.5

Algebraic equation and distance from a point to a line

If the point R belongs to the line, the vector PR and the direction vector v are proportional and commute:

$$PR\, v = v\, PR \quad \Leftrightarrow \quad PR\, v - v\, PR = 0 \quad \Leftrightarrow$$

$$PR \wedge v = 0 \quad \Leftrightarrow \quad PR = v^{-1}\, PR\, v$$

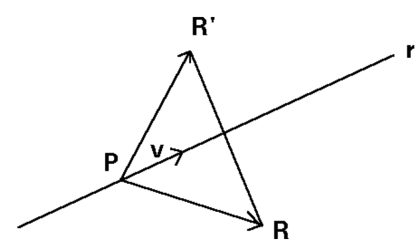


Figure 5.6

The last equation is the *algebraic equation* and shows that the vector PR remains invariant under a reflection in the direction of the line. That is, the point R only belongs to the line when it coincides with the point reflected in this line. Separating components, the *two-point equation* of the line arises:

$$\frac{x - x_P}{v_1} = \frac{y - y_P}{v_2} \quad \Leftrightarrow \quad \frac{x - x_P}{x_Q - x_P} = \frac{y - y_P}{y_Q - y_P}$$

[3] In the book *Geometria Axiomàtica* (Institut d'Estudis Catalans, Barcelona, 1993), Agustí Reventós defines a bijection between the points of a line and the real numbers preserving the order relation as a simplifying axiom of the foundations of geometry. I use implicitly this axiom in this book.

where $v = (v_1, v_2)$.

If the point R does not belong to the line, the reflected point R' differs from R. This allows to calculate the distance from R to the line as the half of the distance between R and R' (figure 5.6). The vector going from R to R' is equal to $PR' - PR$.

$$d(R, r) = \frac{1}{2} \left| R' - R \right| = \frac{1}{2} \left| PR' - PR \right| = \frac{1}{2} \left| v^{-1} PR\, v - PR \right|$$

We will obtain an easier expression for the distance by extracting the direction vector as common factor:

$$d(R, r) = \frac{1}{2} \left| v^{-1} (PR\, v - v\, PR) \right| = \frac{1}{2} \left| v^{-1} \right| \left| PR\, v - v\, PR \right| = \frac{\left| PR \wedge v \right|}{\left| v \right|}$$

In this formula, the distance from R to the line r is the height of the parallelogram formed by the direction vector and (figure 5.7).

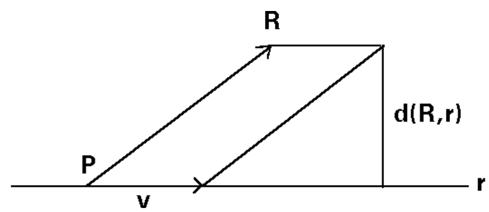


Figure 5.7

Similar line equations can also be deduced using the normal (perpendicular) vector n. Like for the direction vector, the normal vector is not unique because any other proportional vector may also be a normal vector. The normal vector anticommutes with the direction vector of the line, and therefore with the vector PR, P being a given point and R a general point on the line:

$$n\, PR = -PR\, n \quad \Leftrightarrow \quad n\, PR + PR\, n = 0 \quad \Leftrightarrow \quad n \cdot PR = 0 \quad \Leftrightarrow \quad PR = -\, n^{-1}\, PR\, n$$

The last equation is also called the *algebraic equation* and shows that the vector PR is reversed under a reflection in the perpendicular of the line. Taking components, the *general equation* of the line is obtained:

$$n_1\, (x - x_P) + n_2\, (y - y_P) = 0 \qquad \Leftrightarrow \qquad n_1\, x + n_2\, y + c = 0$$

where $n = (n_1, n_2)$ and c is a real constant.

When the point R does not lie on the line, the point R' reflected of R in the perpendicular does not belong to the line. Let PR'' be the reversed of PR', or equivalently R'' be the symmetric point of R with regard to P (figure 5.8). Then, the distance from R to the line r is the half of the distance from R to R'':

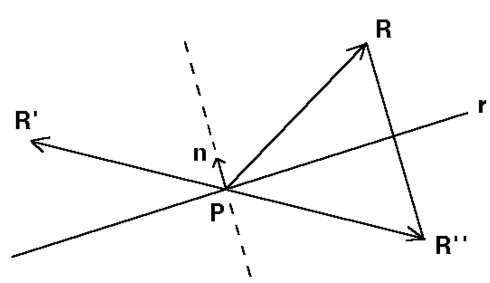


Figure 5.8

$$d(R, r) = \frac{1}{2} \left| PR'' - PR \right|$$

$$=\frac{1}{2}\,|-n^{-1}\,PR\,n-PR\,|=\frac{1}{2}\,|-n^{-1}\,(\,PR\,n+n\,PR\,)\,|=\frac{|\,PR\cdot n\,|}{|\,n\,|}$$

This expression is the modulus of the projection of the vector *PR* upon the perpendicular direction. Both formulas of the distance are equivalent because the normal and direction vectors anticommute:

$$n\,v=-\,v\,n \qquad \Leftrightarrow \qquad n\,v+v\,n=0 \quad \Leftrightarrow \qquad n\cdot v=0$$

Written with components:

$$n_1\,v_1+n_2\,v_2=0$$

From each vector the other one is easily obtained by exchanging components and altering a sign:

$$n=(\,n_1\,,\,n_2\,)=(\,v_2\,,\,-v_1\,)$$

As an application, let us calculate the equation of the line *r* passing through the points (2, 3) and (7, 6). A direction vector is the segment having as extremes both points:

$$v=(7,6)-(2,3)=(5,3)$$

and an equation for this line is:

$$\frac{x-2}{5}=\frac{y-3}{3}$$

From the direction vector we calculate the perpendicular vector *n* and the general equation of the line:

$$n=(3,-5) \qquad\qquad 3\,(x-2)-5\,(y-3)=0 \qquad \Leftrightarrow \qquad 3\,x-5\,y+9=0$$

Now, let us calculate the distance from the point $R=(-2,1)$ to the line *r*. We choose the point (2, 3) as the point *P* on the line:

$$PR=R-P=(-2,1)-(2,3)=(-4,-2)$$

$$d(\,(-2,1),r\,)=\frac{|\,PR\wedge v\,|}{|\,v\,|}=\frac{|\,PR\cdot n\,|}{|\,n\,|}=\frac{|-4\cdot 3+(-2)\cdot(-5)|}{\sqrt{34}}=\sqrt{\frac{2}{17}}$$

For another point $R=(-3,0)$, we will find a null distance indicating that this point lies on *r*.

Slope and intercept equations of a line

If the ordinate is written as a function of the abscissa, the equation so obtained is the *slope-intercept equation*:

$$y = m\,x + b$$

Note that this expression cannot describe the vertical lines whose equation is x = constant. The coefficient m is called the *slope* because it is a measure of the inclination of the line. For any two points P and Q on the line we have:

$$y_P = m\,x_P + b$$

$$y_Q = m\,x_Q + b$$

Subtracting both equations:

$$y_Q - y_P = m\,(x_Q - x_P)$$

we see that the slope is the quotient of ordinate increment divided by the abscissa increment (figure 5.9):

$$m = \frac{y_Q - y_P}{x_Q - x_P}$$

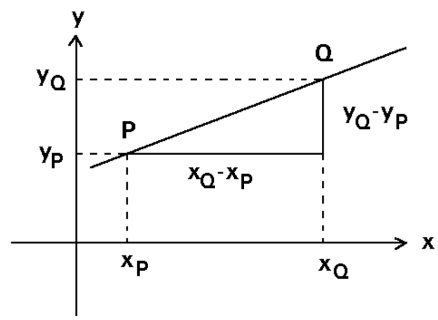


Figure 5.9

This quotient is also the trigonometric tangent of the oriented angle between the line and the positive abscissa semiaxis. For angles larger than $\pi/2$ the slope becomes negative.

b is the ordinate intercepted at the origin, the *y-intercept*:

$$x = 0 \quad \Rightarrow \quad y = b$$

If we know the slope and a point on a line, the *point-slope equation* may be used:

$$y - y_P = m\,(x - x_P)$$

Also, one may write for the *intercept equation* of a line:

$$\frac{x}{a} + \frac{y}{b} = 1$$

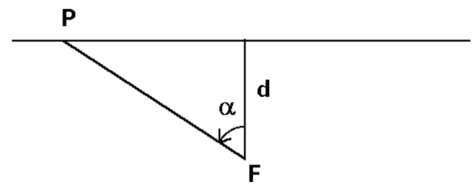
where a and b are the *x-intercept* and *y-intercept* of the line with each coordinate axis.

Polar equation of a line

In this equation, the distance from a fixed point F to the generic point P on the line is a function of the angle α with regard to the perpendicular direction (figure 5.10). If d is the distance from F to the line, then:

$$|FP| = \frac{d}{\cos\alpha}$$

Figure 5.10



This equation allows to relate easily the straight line with the circle and other conic sections.

Intersection of two lines and pencil of lines

The calculation of the intersection of two lines is a very usual problem. Let us suppose that the first line is given by the point *P* and the direction vector *u* and the second one by the point *Q* and the direction vector *v*. Denoting by *R* the intersection point, which belongs to both lines, we have:

$$R = P + k\,u = Q + l\,v \qquad \Leftrightarrow \qquad k\,u - l\,v = Q - P = PQ$$

In order to find the coefficients *k* and *l*, the vector *PQ* must be resolved into a linear combination of *u* and *v*, what results in:

$$k = PQ \wedge v\,(u \wedge v)^{-1} \qquad l = -\,u \wedge PQ\,(u \wedge v)^{-1}$$

The intersection point is also obtained by directly solving the system of the general equations of both lines:

$$\begin{cases} n_1\,x + n_2\,y + c = 0 \\ n_1'\,x + n_2'\,y + c' = 0 \end{cases}$$

The *pencil of lines* passing through this point is the set of lines whose equations are linear combinations of the equations of both given lines:

$$(1-p)\,[\,n_1\,x + n_2\,y + c\,] + p\,[\,n'_1\,x + n'_2\,y + c'\,] = 0$$

where p is a real parameter and $-\infty \le p \le \infty$. Each line of the pencil determines a unique value of p, independently of the fact that the general equation for a line is not unique

If the equation system has a unique point as solution, all the lines of the pencil are concurrent at this point. But in the case of an incompatible system, all the lines of the pencil are parallel, that is, they

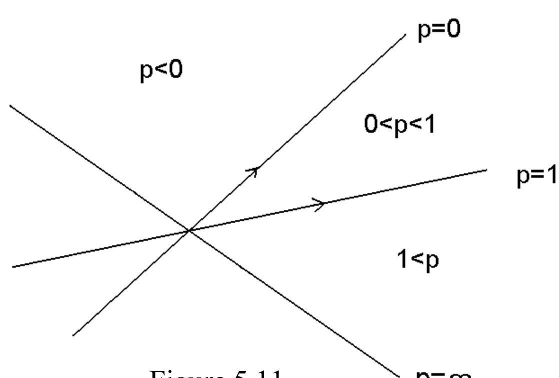


Figure 5.11

intersect in a point at infinity, which gives generality to the concept of pencil of lines.

If the common point R is known or given, then the equation of the pencil is written as:

$$[\,(1-p)\,n_1 + p\,n'_1\,]\,(x - x_R) + [\,(1-p)\,n_2 + p\,n'_2\,]\,(y - y_R) = 0$$

For $0<p<1$, the normal and direction vectors of the p-line are respectively comprised between the normal and direction vectors of both given lines[4]. In the other case the vectors are out of this region (figure 5.11)

For example, calculate the equation of the line passing through the point (3, 4) and the intersection of the lines $3\,x + 2\,y + 4 = 0$ and $2\,x - y + 3 = 0$. The equation of the pencil of lines is:

$$(1-p)\,(3\,x + 2\,y + 4) + p\,(2\,x - y + 3) = 0$$

The line of this pencil to which the point (3, 4) belongs must fulfil:

$$(1-p)\,(3\cdot 3 + 2\cdot 4 + 4) + p\,(2\cdot 3 - 4 + 3) = 0$$

yielding $p = 21/16$ and the line:

$$27\,x - 31\,y + 43 = 0$$

Which is the meaning of the coefficient of linear combination p? Let us write the pencil of lines P determined by the lines R and S as:

$$P = (1-p)\,R + p\,S$$

which implies the same relation for the normal vectors:

$$n_P = (1-p)\,n_R + p\,n_S$$

Taking outer products we obtain:

$$\frac{n_P \wedge n_R}{n_P \wedge n_S} = \frac{p}{p-1} \qquad \Rightarrow \qquad p = \frac{n_P \wedge n_R}{n_P \wedge (n_R - n_S)}$$

These equalities are also valid for direction vectors whenever they have the same modulus than the normal vectors. When the modulus of the normal vectors of the lines R and S are equal we can simplify:

$$\frac{p}{1-p} = \frac{\sin(RP)}{\sin(PS)}$$

In this case, the value $p = 1/2$ corresponds to the bisector line of R and S. We see from this expression and the foregoing ones that each value of p corresponds to a unique line.

---

[4] This statement requires that the angle from the direction vector to the normal vector be a positive right angle.

Dual coordinates

The *duality principle* states that any theorem relating incidence of lines and points implies a dual counterpart where points and lines have been exchanged. The phrase «two points determine a unique line passing through these points» has the dual statement «two lines determine a unique point, intersection of these lines». All the geometric facts have an algebraic counterpart, and the duality is not an exception. With the barycentric coordinates every point R on the plane can be written as linear combination of three non aligned points O, P and Q:

$$R = (1 - p - q)\, O + p\, P + q\, Q = (p, q)$$

where p and q are the coordinates and O the origin.

These three points determine three non parallel lines A, B, and C in the following way:

$$A = \overline{PQ} \qquad B = \overline{QO} \qquad C = \overline{OP}$$

Then the direction vectors $v_A = PQ$, $v_B = OP$, $v_C = QO$ are related by:

$$v_A + v_B + v_C = 0$$

Note that v_B and $-v_C$ are the base of the vectorial plane.

Any line D on the plane can be written as linear combination of these lines:

$$D = (1 - b - c)\, A + b\, B + c\, C$$

This means that the general equation (with point coordinates) of D is a linear combination of the general equations of the lines A, B and C. I call b and c the *dual coordinates* of the line D. In order to distinguish them from point coordinates, I shall write $D = [b, c]$. The choice of the equation for each line must be unambiguous and therefore the normal vector in the implicit equations for A, B and C will be obtained by turning the direction vector over $\pi/2$ counterclockwise.

Let us see some special cases. If the dual coordinate b is zero we have:

$$D = (1 - c)\, A + c\, C$$

which is the equation of the pencil of lines passing through the intersection of the lines A and C, that is, the point P. Then $P = [0, c]$ (for every c). Analogously, $c = 0$ determines the pencil of lines passing through the intersection of the lines A and B, which is the point Q, and then $Q = [b, 0]$ (for every b). The origin of coordinates is the intersection of the lines B and C and then $O = [b, 1 - b]$ (for every b). Compare the dual coordinates of these points:

$$O = [b, 1-b] \qquad P = [0, c] \qquad Q = [b, 0] \qquad \forall\, b, c$$

with the point coordinates of the lines A, B and C:

$$A = (p, 1-p) \qquad B = (0, q) \qquad C = (p, 0) \qquad \forall\, p, q$$

Let us see an example: calculate the dual Cartesian coordinates of the line $2x + 3y + 4 = 0$. The points of the Cartesian base are $O = (0, 0)$, $P = (1, 0)$ and $Q = (0, 1)$. Then the lines of the Cartesian base are A: $-x - y + 1 = 0$, B: $x = 0$, C: $y = 0$. We must solve the identity:

$$2\,x + 3\,y + 4 \equiv a'\,(\,-x - y + 1\,) + b'\,x + c'\,y \qquad \forall\, x, y$$

$$x\,(\,2 + a' - b'\,) + y\,(\,3 + a' - c'\,) + 4 - a' \equiv 0$$

whose solution is:

$$a' = 4 \qquad b' = 6 \qquad c' = 7$$

Dividing by the sum of the coefficients we obtain:

$$\frac{2\,x + 3\,y + 4}{17} \equiv \frac{4}{17}\left(-x - y + 1\right) + \frac{6}{17}\,x + \frac{7}{17}\,y$$

from where the dual coordinates of this line are obtained as $[b, c] = [6/17, 7/17]$. Let us see their meaning. The linear combination of both coordinates axes is a line of the pencil of lines passing through the origin:

$$\frac{6}{13}\,x + \frac{7}{13}\,y = 0 \qquad \text{or} \qquad 6\,x + 7\,y = 0$$

This line intersects the third base line $-x - y + 1 = 0$ at the point $(7, -6)$, whose pencil of lines is described by:

$$a\,(\,-x - y + 1) + (1 - a)\left(\frac{6}{13}\,x + \frac{7}{13}\,y\right) = 0$$

Then $2\,x + 3\,y + 4 = 0$ is the line of this pencil determined by $a = 4/17$.

On the other hand, how may we know whether three lines are concurrent and belong to the same pencil or not? The answer is that the determinant of the dual coordinates must be zero:

$$D = (\,1 - b - c\,)\,A + b\,B + c\,C$$

$$E = (\,1 - b' - c'\,)\,A + b'\,B + c'\,C$$

$$F = (\,1 - b'' - c''\,)\,A + b''\,B + c''\,C$$

$$D, E \text{ and } F \text{ concurrent} \quad \Leftrightarrow \quad \begin{vmatrix} 1 - b - c & b & c \\ 1 - b' - c' & b' & c' \\ 1 - b'' - c'' & b'' & c'' \end{vmatrix} = 0$$

When it happens, we will say that the lines are linearly dependent, and we can write anyone of them as linear combination of the others:

$$F = (1 - k)\, D + k\, E$$

We can express a point with an equation for dual coordinates in the same manner as we express a line with an equation for point coordinates. For example, the point (5, 3) is the intersection of the lines $x = 5$, $y = 3$ whose dual coordinates we calculate now:

$$x - 5 \equiv a'(1 - x - y) + b'x + c'y \qquad \Rightarrow \qquad a' = -5 \quad b' = -4 \quad c' = -5$$

From where a = 5/14, b = 4/14 and c = 5/14. The dual coordinates of x = 5 are [4/14, 5/14]. Analogously:

$$y - 3 \equiv a'(1 - x - y) + b'x + c'y \qquad \Rightarrow \qquad a' = -3 \quad b' = -3 \quad c' = -2$$

From where $a = b$ = 3/8, c = 2/8. The dual coordinates of y = 3 are [3/8, 2/8]. The linear combinations of both lines are the pencil of the point (5, 3), which is described by the parametric equation:

$$[b,\, c] = (1 - k)\left[\frac{4}{14},\, \frac{5}{14}\right] + k\left[\frac{3}{8}, \frac{2}{8}\right]$$

By removing the parameter k, the general equation is obtained:

$$\frac{b - 4/14}{5} = \frac{c - 5/14}{-6} \qquad \Leftrightarrow \qquad 12b + 10c - 7 = 0$$

That is, the *dual direction vector* of the point (5, 3) is [5, –6] and the *dual normal vector* is [6, 5]. In the dual plane, an algebra of dual vectors can be defined. A dual direction vector for a point may be obtained as the difference between the dual coordinates of two lines whose intersection be the point. Then, there exist dual translations of lines:

$$\text{dual} + :\ L \times W \rightarrow L \qquad\qquad L = \{\text{plane lines}\} \qquad W = \{\text{dual vectors}\}$$

$$(A, w) \rightarrow B = A + w$$

That is, we add a fixed dual vector w to the line A in order to obtain another line B.

Let us prove the following theorem: ~~all the points whose dual direction vectors are proportional are aligned with the centroid of the coordinate system .~~

The proof begins from the dual continuous equation for a point P:

$$\frac{b - b_0}{v_1} = \frac{c - c_0}{v_2}$$

which can be written in a parametric form:

$$P: \quad [b,c] = (1-k)[b_0, c_0] + k\,[b_0 + `v_1,\ c_0 + v_2]$$

This equation means that P is the intersection point of the lines with dual coordinates $[b_0, c_0]$ and $[b_0 + v_1,\ c_0 + v_2]$. Then P must be obtained by solving the system of equations of each line for the point coordinates p, q (x, y if Cartesian):

$$\begin{cases} (1 - b_0 - c_0)A + b_0 B + c_0 C = 0 \\ (1 - b_0 - c_0 - v_1 - v_2)A + (b_0 + v_1)B + (c_0 + v_2)C = 0 \end{cases}$$

By subtraction of both equations, we find an equivalent system:

$$\begin{cases} (1 - b_0 - c_0)A + b_0 B + c_0 C = 0 \\ -(v_1 + v_2)A + v_1\, B + v_2\, C = 0 \end{cases}$$

Now, if we consider a set of points with the same (or proportional) dual direction vector, the first equation changes but the second equation remains constant (or proportional). That is, the first line changes but the second line remains constant. Therefore all points will be aligned and lying on the second line:

$$-(v_1 + v_2)A + v_1 B + v_2 C = 0$$

However in which matter do two points differ whether having or not a proportional dual direction vector? We cannot say that two points are aligned, because we need three points at least. Let us search the third point X, rewriting the foregoing equality:

$$\frac{v_1}{v_1 + v_2}(B - A) + \frac{v_2}{v_1 + v_2}(C - A) = 0$$

Now the variation of the components of the dual vector generates the pencil of the lines $B - A$ and $C - A$. That is, the intersection of both lines is the point X searched:

$$X: \quad \begin{cases} B - A = 0 \\ C - A = 0 \end{cases}$$

Then, two points have a proportional dual direction vector if they are aligned with the point X, intersection of the lines $B - A$ and $C - A$. However, note that the addition of coefficients of each line is zero instead of one, so that one dual coordinate of each line is infinite:

$$B - A = [\infty, c] \qquad\qquad C - A = [b,\ \infty]$$

I call X the *point at the dual infinity* or simply the *dual infinity point*. The dual infinity point has finite coordinates and a very well defined position. Then the points with proportional dual vector are always aligned with the dual infinity point, and I shall say that they are *parallel points* in a dual sense, of course. In order to precise which point is X let us take in mind that the lines $B - A$ and $C - A$ are the medians of the triangle OPQ (the point base). Hence the dual infinity point is the centroid of the base triangle *OPQ*. With Cartesian coordinates $O = (0, 0)$, $P = (1, 0)$, $Q = (0, 1)$ and $X = (1/3, 1/3)$. This ends the proof.

Summarising, we can say that two parallel points are aligned with the dual infinity point, which is the dual statement of the fact that two parallel lines meet at the infinity, that is, there is a *line at the infinity*, in the usual sense. This is the novelty of the projective geometry in comparison with the Euclidean geometry. In fact, the problem arises because the point coordinates take infinite values, but the line L at the infinity is a well defined line with finite dual coordinates [1/3, 1/3]:

$$L = \left[\frac{1}{3}, \frac{1}{3}\right] = \frac{A + B + C}{3}$$

This means that the line L at the infinity belongs to the pencil of the lines $(A + B)/2$ and C. But both lines are parallel because their direction vectors are proportional:

$$v_A + v_B = -v_C$$

That is, the lines $(A + B)/2$ and C meet at a point located at an infinite distance from the origin of coordinates. Since any other parallel line meets them also at the infinity, this argument does not suffice. However we can understand that the line L also belongs to the pencil of the lines A and $(B + C)/2$, which are also parallel with another direction, that is, they meet at another point of the infinity. Any other pencil we take has always its point of intersection located at the infinity. Then L only has points located at the infinity and because of this it is called the line at the infinity. Summarising we can say that the line at the infinity is the centroid of the three base lines in spite of the incompatibility of its equation:

$$L = \frac{A + B + C}{3} \qquad \Rightarrow \qquad \frac{(-x - y + 1) + x + y}{3} = 0 \qquad \Rightarrow \qquad \frac{1}{3} = 0$$

Moreover, we may interpret the parallel lines as those lines aligned (in a dual sense) with the line at the infinity:

$$E = [b_E, c_E] \qquad\qquad F = [b_F, c_F]$$

$$E \parallel F \qquad \Leftrightarrow \qquad \frac{b_F - b_E}{c_F - c_E} = \frac{b_F - 1/3}{c_F - 1/3}$$

This is a useful equality because it allows us to know whether two lines are parallel from their dual coordinates.

Many of the incidence theorems (and also their dual theorems) can be solved by means of line equations or dual coordinates. A proper example is the proof of the Desargues theorem.

The Desargues theorem

Given two triangles ABC and $A'B'C'$, let P be the intersection of the prolongation of the side AB with the side $A'B'$, Q the intersection of BC with $B'C'$, and R the intersection of CA with $C'A'$. The points P, Q and R are aligned if and only if the lines AA', BB' and CC' meet at the same point.

Proof $\Rightarrow$ The hypothesis states that the lines AA', BB' and CC' intersect at the same point O (figure 5.12):

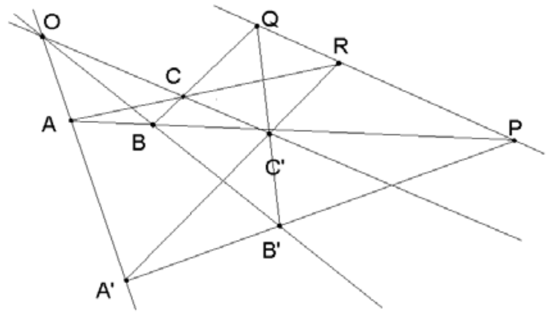


Figure 5.12

$$O = a\,A + (\,1 - a\,)\,A'$$

$$O = b\,B + (\,1 - b\,)\,B'$$

$$O = c\,C + (\,1 - c\,)\,C'$$

with a, b and c being real. Equating the first and second equations we obtain:

$$a\,A + (\,1 - a\,)\,A' = b\,B + (\,1 - b\,)\,B'$$

which can be rearranged as:

$$a\,A - b\,B = -\,(\,1 - a\,)\,A' + (\,1 - b\,)\,B'$$

Dividing by $a - b$ the sum of the coefficients becomes the unity, and then the equation represents the intersection of the lines AB and $A'B'$, which is the point P:

$$P = \frac{a}{a-b}A - \frac{b}{a-b}B = \frac{a-1}{a-b}A' - \frac{b-1}{a-b}B'$$

By equating the second and third equations for the point O, and the third and first ones, analogous equations for Q and R are obtained:

$$Q = \frac{b}{b-c}B - \frac{c}{b-c}C = \frac{b-1}{b-c}B' - \frac{c-1}{b-c}C'$$

$$R = \frac{c}{c-a}C - \frac{a}{c-a}A = \frac{c-1}{c-a}C' - \frac{a-1}{c-a}A'$$

Now we must prove that P, Q and R are aligned, that is, fulfil the equation:

$$R = d\,P + (\,1 - d\,)\,Q \qquad \text{with } d \text{ real}$$

With the substitution of the former equations into the last one we have:

$$\frac{c}{c-a}C - \frac{a}{c-a}A = d\left[\frac{a}{a-b}A - \frac{b}{b-a}B\right] + \left(1 - d\,\right)\left[\frac{b}{b-c}B - \frac{c}{b-c}C\right]$$

Arranging all the terms at the left hand side, the expression obtained must be identical to zero because A, B and C are non aligned points:

$$\left[-\frac{d\,a}{a-b}-\frac{a}{c-a}\right]A+\left[\frac{d\,b}{a-b}-\frac{b(1-d)}{b-c}\right]B+\left[\frac{c}{c-a}+\frac{c(1-d)}{b-c}\right]C\equiv 0$$

This implies that the three coefficients must be null simultaneously yielding a unique value for d,

$$d=\frac{a-b}{a-c}$$

fact which proves the alignment of P, Q and R, and gives the relation for the distances between points:

$$QR\;QP^{-1}=(\,A'O\;\;A'A^{-1}-B'O\;\;B'B^{-1}\,)\,(\,A'O\;\;A'A^{-1}-C'O\;\;C'C^{-1}\,)^{-1}$$

Proof $\Leftarrow$ We will prove the Desargues theorem in the other direction following the same algebraic way but applying the duality, that is, I shall only change the words. *O, A, B, C, P, Q* and *R* will be now lines (figure 5.13).

The hypothesis states that the points AA', BB' and CC' belong to the same line O, that is, O belongs to the pencil of the lines A and A', but also to the pencil of the lines BB' and CC':

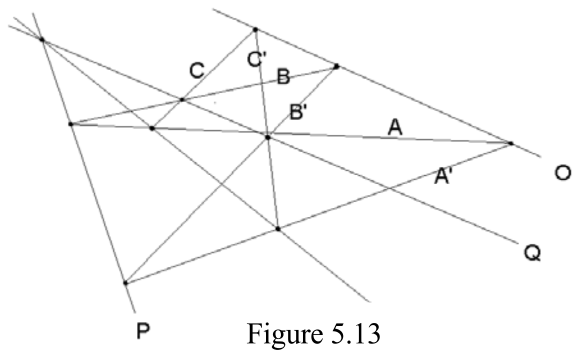


Figure 5.13

$$O=a\,A+(\,1-a\,)\,A'$$

$$O=b\,B+(\,1-b\,)\,B'$$

$$O=c\,C+(\,1-c\,)\,C'$$

with a, b and c being real coefficients. Equating the first and second equations we obtain:

$$a\,A+(\,1-a\,)\,A'=b\,B+(\,1-b\,)\,B'$$

which can be rearranged as:

$$a\,A-b\,B=-(\,1-a\,)\,A'+(\,1-b\,)\,B'$$

Dividing by $a-b$ the sum of the coefficients becomes the unity, and then the equation represents the line passing through the points AB and $A'B'$ (belonging to both pencils), which is the line P:

$$P=\frac{a}{a-b}A-\frac{b}{a-b}=\frac{a-1}{a-b}A'-\frac{b-1}{a-b}B'$$

By equating the second and third equations for the line O, and the third and first ones, analogous equations for Q and R are obtained:

$$Q = \frac{b}{b-c}B - \frac{c}{b-c}C = \frac{b-1}{b-c}B' - \frac{c-1}{b-c}C'$$

$$R = \frac{c}{c-a}C - \frac{a}{c-a}A = \frac{c-1}{c-a}C' - \frac{a-1}{c-a}A'$$

Now we must prove that the lines P, Q and R belong to the same pencil, that is, fulfil the equation:

$$R = d\,P + (\,1 - d\,)\,Q \qquad\qquad \text{with } d \text{ real}$$

With the substitution of the former equations into the last one we have:

$$\frac{c}{c-a}C - \frac{a}{c-a}A = d\left[\frac{a}{a-b}A - \frac{b}{b-a}B\right] + (1-d\,)\left[\frac{b}{b-c}B - \frac{c}{b-c}C\right]$$

Arranging all the terms at the left hand side, the expression obtained must be identical to zero because A, B and C are independent lines (not belonging to the same pencil):

$$\left[-\frac{d\,a}{a-b} - \frac{a}{c-a}\right]A + \left[\frac{d\,b}{a-b} - \frac{b(1-d)}{b-c}\right]B + \left[\frac{c}{c-a} + \frac{c(1-d)}{b-c}\right]C \equiv 0$$

This implies that the three coefficients must be null simultaneously yielding a unique value for d,

$$d = \frac{a-b}{a-c}$$

fact which proves that P, Q and R are lines of the same pencil.

Exercises

5.1 Let A = (2, 4), B = (4, –3) and C = (2, –5) be three consecutive vertices of a parallelogram. Calculate the fourth vertex D and the area of the parallelogram.

5.2 Prove the Euler's theorem: for any four points A, B, C and D the product $AD\,BC$ + $BD\,CA$ + $CD\,AB$ vanishes if and only if A, B and C are aligned.

5.3 Consider a coordinate system with vectors $\{ e_1, e_2 \}$ where $|e_1| = 1$, $|e_2| = 1$ and the angle formed by both vectors is $\pi/3$.
a) Calculate the area of the triangle ABC being A = (2, 2), B = (4, 4), C = (4, 2).
b) Calculate the distance between A and B, B and C, C and A.

5.4 Construct a trapezoid whose sides $|AB|$, $|BC|$, $|CD|$ and $|DA|$ are known and

AB is parallel to *CD*. Study for which values of the sides does the trapezoid exist.

5.5 Given any coordinate system with points $\{O, P, Q\}$ and any point R with coordinates (p, q) in this system, show that:

$$1 - p - q = \frac{Area\ RPQ}{Area\ OPQ} \qquad p = \frac{Area\ ORQ}{Area\ OPQ} \qquad q = \frac{Area\ OPR}{Area\ OPQ}$$

5.6 Let the points $A = (2, 3)$, $B = (5, 4)$, $C = (1, 6)$ be given. Calculate the distance from C to the line AB and the angle between the lines AB and AC.

5.7 Given any barycentric coordinate system defined by the points $\{O, P, Q\}$ and the non aligned transformed points $\{O', P', Q'\}$, an *affinity* (or an *affine transformation*) is defined as that geometric transformation which maps each point D to D' in the following way:

$$D = (1 - x - y)\, O + x\, P + y\, Q \rightarrow \qquad D' = (1 - x - y)\, O' + x\, P' + y\, Q'$$

Prove that:
a) An affinity maps lines onto lines.
b) An affinity is equivalent to a linear mapping of the coordinates, that is, the coordinates of any transformed point are linear functions of the coordinates of the original point.
c) An affinity preserves the coordinates expressed in any other set of independent points different of the given base:

$$D = (1 - b - c)\, A + b\, B + c\, C \rightarrow \qquad D' = (1 - b - c)\, A' + b\, B' + c\, C'$$

d) An affinity maps a parallelogram to another parallelogram, and hence, parallel lines to parallel lines.
e) An affinity preserves the ratio $DE\, DF^{-1}$ for any three aligned points D, E and F.

5.8 If $\{A, B, C\}$ is a base of lines and $\{A', B', C'\}$ their transformed lines -the lines of each set being independent-, consider the geometric transformation which maps each line D to D' in the following way:

$$D = (1 - b - c)\, A + b\, B + c\, C \rightarrow \qquad D' = (1 - b - c)\, A' + b\, B' + c\, C'$$

a) Prove that every pencil of lines is mapped to another pencil of lines.
b) The dual coordinates of D' are linear functions of the dual coordinates of D.
c) This transformation preserves the coefficients which express a line as a linear combination of any three non concurrent lines, that is, in the foregoing mapping $\{A, B, C\}$ have not to be necessarily the dual coordinate base and can be any other set of independent lines.
d) Parallel points are mapped to parallel points.
e) For any three concurrent lines P, Q and R, the single ratio of the dual vectors $PQ\, PR^{-1}$ is preserved.
f) Using the formula of the *cross ratio* of a pencil of any four lines (*ABCD*) as a

function of their direction vectors[5]:

$$(ABCD) = \frac{v_A \wedge v_C \; v_B \wedge v_D}{v_A \wedge v_D \; v_B \wedge v_C}$$

show that it is preserved.

5.9 Calculate the dual coordinates of the lines $x - y + 1 = 0$ and $x - y + 3 = 0$. See that they are aligned in the dual plane with the line at the infinity, whose dual coordinates are [1/3, 1/3], and therefore are parallel.

5.10 Calculate the dual equations of the points (2, 1) and (–3, –1) and their direction vectors. See that they are parallel points. Hence prove that they are aligned with the dual infinity point, the centroid of the coordinate system (1/3, 1/3).

[5] This formula is deduced in the chapter devoted to the cross ratio.