## 6. ANGLES AND ELEMENTAL TRIGONOMETRY

Here, the basic identities of the elemental trigonometry are deduced in close connection with basic geometric facts, a very useful point of view for our pupils.

### Sum of the angles of a polygon

Firstly let us see the special case of a triangle. For any triangle *ABC* the following identity holds:

$$CA\,AB^{-1}\,AB\,BC^{-1}\,BC\,CA^{-1} = 1$$

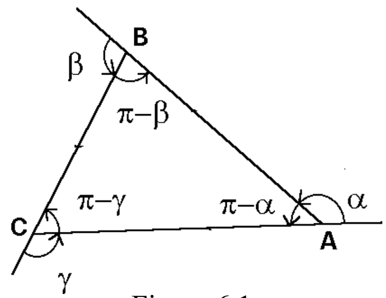

Figure 6.1

Let $\alpha$, $\beta$ and $\gamma$ be the exterior angles between the sides *CA* and *AB*, *AB* and *BC*, *BC* and *CA* respectively (figure 6.1). Applying the definition of geometric quotient, the modulus of all the sides are simplified and only the exponentials of the arguments remain:

$$\exp(\alpha\, e_{12})\exp(\beta\, e_{12})\exp(\gamma\, e_{12}) = \exp[(\alpha+\beta+\gamma)e_{12}] = 1$$

The three angles have the same orientation, which we suppose positive, and are lesser than $\pi$. Hence, since the exponential is equal to the unity, the addition of the three angles must be equal to $2\pi$:

$$\alpha+\beta+\gamma = 2\pi$$

The interior angles, those formed by *AB* and *AC*, *BC* and *CA*, *CA* and *CB* are supplementary of $\alpha$, $\beta$, $\gamma$ (figure 6.1). Therefore the sum of the angles of a triangle is equal to $\pi$:

$$(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma) = \pi$$

This result is generalised to any polygon from the following identity:

$$AB\,BC^{-1}\,BC\ \ldots\ YZ^{-1}\,YZ\,ZA^{-1}\,ZA\,AB^{-1} = 1$$

Let $\alpha$, $\beta$, $\gamma$ ... $\omega$ be the exterior angles formed by the sides *ZA* and *AB*, *AB* and *BC*, *BC* and *CD*, ..., *YZ* and *ZA*, respectively. After the simplification of the modulus of all vectors, we have:

$$\exp[(\alpha+\beta+\gamma+\ldots+\omega)e_{12}] = 1$$

Let us suppose that the orientation defined by the vertices *A*, *B*, *C* ... *Z* is counterclockwise, although the exterior angles be not necessarily all positive[1]. Translating

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[1] That is, it is not needed that the polygon be convex.

them to a common vertex, each angle is placed close each other following the order of the perimeter and summing one turn:

$$\alpha + \beta + \gamma + ... + \omega = 2\pi$$

The interior angles formed by the sides *AB* and *ZA*, *BC* and *BA*, *CD* and *CB*,... *ZA* and *ZY* are supplementary of $\alpha$, $\beta$, $\gamma$, ... $\omega$. Therefore the sum of the angles of a polygon is:

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) + ... + (\pi - \omega) = n\,\pi - 2\pi = (n - 2)\pi$$

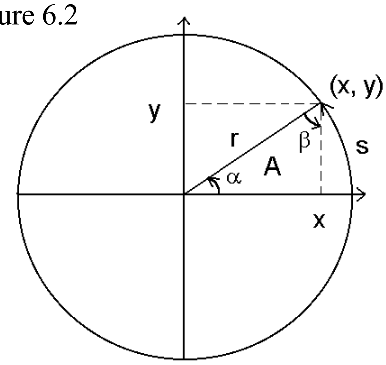
*n* being the number of sides of the polygon. The deduction for the clockwise orientation of the polygon is analogous with the only difference that the result is negative.

### Definition of trigonometric functions and fundamental identities

Let us consider a circle with radius *r* (figure 6.2). The extreme of the radius is a point on the circumference with coordinates (*x*, *y*). The arc between the positive *X* semiaxis and this point (*x*, *y*) has an oriented length *s*, positive if counterclockwise and otherwise negative. Also, the *X*-axis, the arc of circumference and the radius delimit a sector with an oriented area *A*. An *oriented angle* $\alpha$ is defined as the quotient of the arc length divided by the radius[2]:

$$\alpha = \frac{s}{r}$$

Figure 6.2




Since the area of the sector is proportional to the arc length and the area of the circle is $2\pi r$, it follows that:

$$A = \frac{\alpha\, r^2}{2} \quad \Leftrightarrow \quad \alpha = \frac{2\,A}{r^2}$$

The *trigonometric functions*[3], *sine*, *cosine* and *tangent* of the angle $\alpha$, are respectively defined as the ratios:

$$\sin\alpha = \frac{y}{r} \qquad \cos\alpha = \frac{x}{r} \qquad \mathrm{tg}\,\alpha = \frac{y}{x}$$

and the *cosecant*, *secant* and *cotangent* as their inverse fractions:

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[2] With this definition it is said that the angle is given in *radians*, although an angle is a quotient of lengths and therefore a number without dimensions.

[3] Also called *circular functions* due to obvious reasons, to be distinguished from the *hyperbolic functions*.

$$\csc\alpha = \frac{r}{y} \equiv \frac{1}{\sin\alpha} \qquad \sec\alpha = \frac{r}{x} \equiv \frac{1}{\cos\alpha} \qquad \cot\alpha = \frac{x}{y} \equiv \frac{1}{\mathrm{tg}\,\alpha}$$

being $r$ related with $x$ and $y$ by the Pythagorean theorem:

$$r^2 = x^2 + y^2$$

Then the radius vector $v$ is:

$$v = x\, e_1 + y\, e_2 = r\,(\cos\alpha\; e_1 + \sin\alpha\; e_2)$$

From these definitions the fundamental identities follow:

$$\mathrm{tg}\,\alpha \equiv \frac{\sin\alpha}{\cos\alpha} \qquad \sin^2\alpha + \cos^2\alpha \equiv 1 \qquad 1 + \mathrm{tg}^2\,\alpha \equiv \frac{1}{\cos^2\alpha} \equiv \sec^2\alpha$$

If we take the opposite angle $-\alpha$ instead of $\alpha$, the sign of $y$ is changed while $x$ and $r$ are preserved, so we obtain the parity relations:

$$\sin(-\alpha) \equiv -\sin\alpha \qquad \cos(-\alpha) \equiv \cos\alpha \qquad \mathrm{tg}(-\alpha) \equiv -\,\mathrm{tg}\,\alpha$$

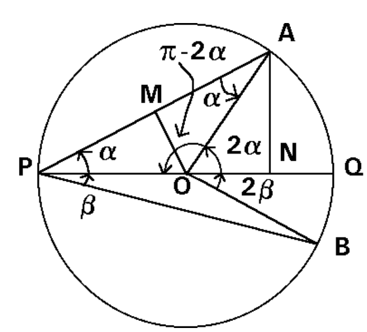
The sine and tangent are odd functions while the cosine is an even function.

Look at the figure 6.2: $\beta = \pi/2 - \alpha$ is the *complementary* angle of $\alpha$. If $\alpha$ is higher than $\pi/2$, the angle $\beta$ becomes negative. On the other hand, if the angle $\alpha$ is negative then $\beta$ is higher than $\pi/2$. The trigonometric ratios for $\beta$ give the complementary angle identities:

$$\sin\beta = \frac{x}{r} \equiv \cos\alpha \qquad \cos\beta = \frac{y}{r} \equiv \sin\alpha \qquad \mathrm{tg}\,\beta = \frac{x}{y} \equiv \frac{1}{\mathrm{tg}\,\alpha}$$

**Angle inscribed in a circle and double angle identities**

Figure 6.3




Let us draw any diameter $PQ$ and any radius $OA$ (figure 6.3) in a circle with centre $O$. Since $OP$ is also a radius, the triangle $POA$ is isosceles and the angles $OPA$ and $PAO$, which will be denoted as $\alpha$, are equal. Because the addition of the three angles is equal to $\pi$, the angle $AOP$ is $\pi - 2\alpha$. The angle $QOA$ is supplementary of $AOP$, and therefore is equal to $2\alpha$, the double of the angle $APQ$. Let us draw from $A$ a segment perpendicular to the diameter and touching it at the point $N$. By the definition of sine we have:

$$\sin 2\alpha = \frac{|NA|}{|OA|} = \frac{|NA|}{|PA|}\,\frac{|PA|}{|OA|}$$

The first quotient is $\sin\alpha$ for the triangle *PNA*. If *M* is the midpoint of the segment *PA*, then $|PA| = 2\ |MA|$ and the second quotient is equal to 2 $\cos\alpha$ for the triangle *MOA*:

$$\sin 2\alpha = 2\sin\alpha \frac{|MA|}{|OA|} \equiv 2\sin\alpha\ \cos\alpha$$

Through an analogous way we obtain $\cos 2\alpha$:

$$\cos 2\alpha = \frac{|ON|}{|OA|} = \frac{\left(|PN| - |PO|\right)}{|PA|}\frac{|PA|}{|OA|} \equiv 2\cos^2\alpha - 1 \equiv \cos^2\alpha - \sin^2\alpha$$

Also this result may be obtained from the second fundamental identity. In order to obtain the tangent of the double angle we make use of the first fundamental identity:

$$\operatorname{tg} 2\alpha \equiv \frac{\sin 2\alpha}{\cos 2\alpha} \equiv \frac{2\sin\alpha\ \cos\alpha}{\cos^2\alpha - \sin^2\alpha} \equiv \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$$

Finally, let us draw any other segment *PB* (figure 6.3). The angle *BPQ* will be denoted as $\beta$. By the same arguments as above the angle *BOQ* is $2\beta$. While the angle *BPA* is $\alpha + \beta$, the angle *BOA* is $2\alpha + 2\beta$: an angle inscribed in a circumference is equal to the half of the central angle (angle whose vertex is the centre of the circumference) which intercepts the same arc of circle. Consequently, all the angles inscribed in the same circle and intercepting the same arc are equal independently of the position of the vertex on the circle.

### Addition of vectors and sum of trigonometric functions

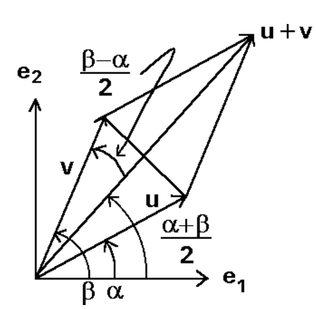
Let us consider two unitary vectors forming the angles $\alpha$ and $\beta$ with the $e_1$ direction (figure 6.4):

$$u = e_1 \cos\alpha + e_2 \sin\alpha \qquad\qquad v = e_1\cos\beta + e_2\sin\beta$$

$$u + v = e_1\left(\cos\alpha + \cos\beta\right) + e_2\left(\sin\alpha + \sin\beta\right)$$

The addition of both vectors, $u + v$, is the diagonal of the rhombus which they form, whence it follows that the long diagonal is the bisector of the angle $\alpha - \beta$ between both vectors. The short diagonal cuts the long diagonal perpendicularly forming four right triangles. Then the modulus of $u + v$ is equal to the double of the cosine of the half of this angle:

Figure 6.4



$$|u+v| = 2\cos\frac{\alpha-\beta}{2}$$

Moreover, the addition vector forms an angle $(\alpha+\beta)/2$ with the $e_1$ direction:

$$u+v = 2\cos\frac{\alpha-\beta}{2}\left(e_1\cos\frac{\alpha+\beta}{2} + e_2\sin\frac{\alpha+\beta}{2}\right)$$

By identifying this expression for $u+v$ with that obtained above, we arrive at two identities, one for each component:

$$\cos\alpha + \cos\beta \equiv 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha + \sin\beta \equiv 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

In a similar manner, but using a subtraction of unitary vectors, the other pair of identities are obtained:

$$\cos\alpha - \cos\beta \equiv -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta \equiv 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

The addition and subtraction of tangents are obtained through the common denominator:

$$\operatorname{tg}\alpha \pm \operatorname{tg}\beta \equiv \frac{\sin\alpha}{\cos\alpha} \pm \frac{\sin\beta}{\cos\beta} \equiv \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \equiv \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$

**Product of vectors and addition identities**

Let us see at the figure 6.4, but now we calculate the product of both vectors:

$$v\,u = (e_1\cos\beta + e_2\sin\beta)(e_1\cos\alpha + e_2\sin\alpha) =$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta + e_{12}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

Since $u$ and $v$ are unitary vectors, their product is a complex number with unitary modulus and argument equal to the oriented angle between them:

$$v\,u = \cos(\alpha-\beta) + e_{12}\sin(\alpha-\beta)$$

The identification of both equations gives the trigonometric functions of the angles difference:

$$\sin(\alpha-\beta)\equiv\sin\alpha\,\cos\beta-\cos\alpha\,\sin\beta$$

$$\cos(\alpha-\beta)\equiv\cos\alpha\cos\beta+\sin\alpha\,\sin\beta$$

Taking into account that the sine and the cosine are odd and even functions respectively, one obtains the identities for the angles addition:

$$\sin(\alpha+\beta)\equiv\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta$$

$$\cos(\alpha+\beta)\equiv\cos\alpha\cos\beta-\sin\alpha\,\sin\beta$$

### Rotations and De Moivre's identity

If $v'$ is the result of turning the vector $v$ over an angle $\alpha$ then:

$$v'=v\left(\cos\alpha+e_{12}\sin\alpha\right)$$

To repeat a rotation of angle $\alpha$ by $n$ times is the same thing as to turn over an angle $n\alpha$:

$$v''=v\left(\cos\alpha+e_{12}\sin\alpha\right)^n=v\left(\cos n\alpha+e_{12}\sin n\alpha\right)$$

$$\Rightarrow\qquad \cos n\alpha+e_{12}\sin n\alpha\equiv\left(\cos\alpha+e_{12}\sin\alpha\right)^n$$

This is the De Moivre's identity[4], which allows to calculate the trigonometric functions of multiplies of a certain angle through the binomial theorem. For example, for $n = 3$ we have:

$$\cos 3\alpha+e_{12}\sin 3\alpha\equiv\left(\cos\alpha+e_{12}\sin\alpha\right)^3$$

$$\equiv\cos^3\alpha-3\cos\alpha\,\sin^2\alpha+e_{12}\left(3\cos^2\alpha\sin\alpha-\sin^3\alpha\right)$$

Splitting the real and imaginary parts one obtains:

$$\cos 3\alpha\equiv\cos^3\alpha-3\cos\alpha\,\sin^2\alpha \qquad\qquad \sin 3\alpha\equiv 3\cos^2\alpha\,\sin\alpha-\sin^3\alpha$$

And dividing both identities one arrives at:

$$\mathrm{tg}\,3\alpha\equiv\frac{3\,\mathrm{tg}\,\alpha-\mathrm{tg}^3\,\alpha}{1-3\,\mathrm{tg}^2\,\alpha}$$

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[4] With the Euler's identity, the De Moivre's identity is:

$$\exp(n\alpha\, e_{12})\equiv\left[\exp(\alpha\, e_{12})\right]^n$$

**Inverse trigonometric functions**

The arcsine, arccosine and arctangent are defined as the inverse functions of the sine, cosine and tangent. They are multivalued functions and the principal values are taken in the following intervals:

$$y = \arcsin x \quad \Leftrightarrow \quad x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = \arccos x \quad \Leftrightarrow \quad x = \cos y \quad \text{and} \quad 0 \le y \le \pi$$

$$y = \operatorname{arctg} x \quad \Leftrightarrow \quad x = \operatorname{tg} y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \operatorname{arccot} x \quad \Leftrightarrow \quad x = \cot y \quad \text{and} \quad 0 < y < \pi$$

From the definitions the parity of the inverse functions follow immediately:

$$\arcsin x \equiv -\arcsin(-x) \qquad \arccos x \equiv \arccos(-x)$$

$$\operatorname{arctg} x \equiv -\operatorname{arctg}(-x)$$

Through the fundamental identities for the circular functions one obtains the following identities[5] for the inverse functions:

$$\arcsin x \equiv \left[\arccos\sqrt{1-x^2}\right] \equiv \operatorname{arctg}\frac{x}{\sqrt{1-x^2}}$$

$$\arccos x \equiv \left[\arcsin\sqrt{1-x^2}\right] \equiv \left[\operatorname{arctg}\frac{\sqrt{1-x^2}}{x}\right]$$

$$\operatorname{arctg} x \equiv \arcsin\frac{x}{\sqrt{1+x^2}} \equiv \left[\arccos\frac{1}{\sqrt{1+x^2}}\right]$$

where the brackets indicate that the identity only holds for positive values.

From the complementary angles identities we have:

$$\arcsin x \equiv \frac{\pi}{2} - \arccos x \qquad \operatorname{arctg} x \equiv \frac{\pi}{2} - \operatorname{arc\,cot} x$$

[5] The only inverse circular function predefined in the language Basic is the arctangent ATN(X), so these identities allows us to program the arcsine and arccosine.

**Exercises**

6.1 Prove the law of sines, cosines and tangents: if $a$, $b$ and $c$ are the sides of a triangle respectively opposite to the angles $\alpha$, $\beta$ and $\gamma$, then:

$$\frac{|a|}{\sin\alpha}=\frac{|b|}{\sin\beta}=\frac{|c|}{\sin\gamma} \qquad \text{(law of sines)}$$

$$c^2=a^2+b^2-2\;|a|\;|b|\;\cos\gamma \qquad \text{(law of cosines)}$$

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\operatorname{tg}\dfrac{\alpha+\beta}{2}}{\operatorname{tg}\dfrac{\alpha-\beta}{2}} \qquad \text{(law of tangents)}$$

Hint: use the inner and outer products of sides.

6.2 Prove the following trigonometric identities:

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-a)\equiv 4\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}-1$$

$$\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)\equiv -4\sin\frac{\alpha-\beta}{2}\,\sin\frac{\beta-\gamma}{2}\,\sin\frac{\gamma-\alpha}{2}$$

6.3 Express the trigonometric functions of $4\alpha$ as a polynomial of the trigonometric functions of $\alpha$.

6.4 Let $P$ be a point on a circle arc whose extremes are the points $A$ and $B$. Prove that the sum of the chords $AP$ and $PB$ is maximal when $P$ is the midpoint of the arc $AB$.

6.5 Prove the Mollweide's formulas for a triangle:

$$\frac{|a|+|b|}{|c|}=\frac{\cos\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\gamma}{2}} \qquad\qquad \frac{|a|-|b|}{|c|}=\frac{\sin\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\gamma}{2}}$$

6.6 Deduce also the projection formulas:

$$|a|=|b|\cos\gamma+|c|\cos\beta \qquad |b|=|c|\cos\alpha+|a|\cos\gamma \qquad |c|=|a|\cos\beta+|b|\cos\alpha$$

6.7 Prove the half angle identities:

$$\sin\frac{\alpha}{2}\equiv\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2}\equiv\pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \operatorname{tg}\frac{\alpha}{2}\equiv\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\equiv\frac{1-\cos\alpha}{\sin\alpha}\equiv\frac{\sin\alpha}{1+\cos\alpha}$$

## 7. SIMILARITIES AND SINGLE RATIO

Two geometric figures are *similar* if they have the same shape. If the orientation of both figures is the same, they are said to be *directly* similar. For example, the dials of clocks are directly similar. On the other hand, two figures can have the same shape but different orientation. Then they are said to be *oppositely* similar. For example our hands are oppositely similar. However these intuitive concepts are insufficient and the similarity must be defined with more precision.

### Direct similarity (similitude)

If two vectors $u$ and $v$ on the plane form the same angle as the angle between the vectors $w$ and $t$, and they have proportional lengths (figure 7.1), then they are geometrically proportional:

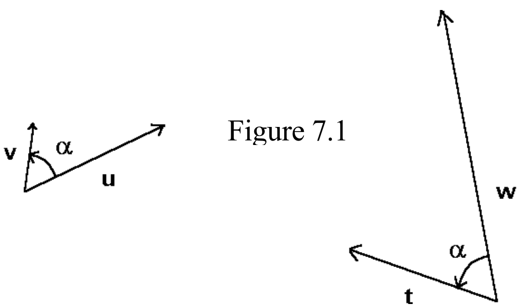



Figure 7.1

$$\left.\begin{array}{c}\alpha(u,v)=\alpha(w,t)\\ \dfrac{|u|}{|v|}=\dfrac{|w|}{|t|}\end{array}\right\} \Rightarrow \quad u\,v^{-1}=w\,t^{-1}$$

that is, the geometric quotient of $u$ and $v$ is equal to the quotient of $w$ and $t$, which is a complex number. This definition of a geometric quotient is also valid for vectors in the space provided that the four vectors lie on the same plane. In this case, the geometric quotient is a quaternion, as Hamilton showed.

The geometric proportionality for vectors allows to define the similarity of triangles. Two triangles $ABC$ and $A'B'C'$ are said to be *directly similar* and their vertices and sides denoted with the same letters are *homologous* if:

$$AB\,BC^{-1}=A'B'\,B'C'^{\,-1}$$

that is, if the geometric quotient of two sides of the first triangle is equal to the quotient of the homologous sides of the second triangle. Arranging the vectors of this equation one obtains the equality of the quotients of the homologous sides:

$$AB^{-1}\,A'B'=BC^{-1}\,B'C'$$

One can prove easily that the third quotient of homologous sides also coincides with the other quotients:

$$AB^{-1}\,A'B'=BC^{-1}\,B'C'=CA^{-1}\,C'A'=r$$

The *similarity ratio* $r$ is defined as the quotient of every pair of homologous sides, which is a complex constant. The modulus of the similarity ratio is the size ratio and the argument is the angle of rotation of the triangle $A'B'C'$ with respect to the triangle $ABC$.

$$r = \frac{|A'B'|}{|AB|} \exp[\alpha(AB, A'B') e_{12}]$$

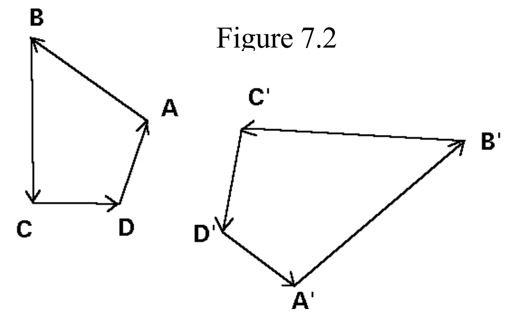


Figure 7.2

The definition of similarity is generalised to any pair of polygons in the following way. Let the polygons *ABC...Z* and *A'B'C'...Z'* be. They are said to be directly similar with similarity ratio $r$ and the sides denoted with the same letters to be homologous if:

$$r = AB^{-1}\, A'B' = BC^{-1}\, B'C' = CD^{-1}\, C'D' = \ldots = YZ^{-1}\, Y'Z' = ZA^{-1}\, Z'A'$$

One of these equalities depends on the others and we do not need to know whether it is fulfilled. Here also, the modulus of $r$ is the size ratio of both polygons and the argument is the angle of rotation. The fact that the homologous exterior and interior angles are equal for directly similar polygons (figure 7.2) is trivial because:

$$B'A'\, B'C'^{-1} = BA\, BC^{-1} \qquad \Rightarrow \qquad \text{angle } A'B'C' = \text{angle } ABC$$

$$C'B'\, C'D'^{-1} = CB\, CD^{-1} \qquad \Rightarrow \qquad \text{angle } B'C'D' = \text{angle } BCD \qquad \text{etc.}$$

The direct similarity is an equivalence relation since it has the reflexive, symmetric and transitive properties. This means that there are classes of equivalence with directly similar figures.

A similitude with $|r|=1$ is called a *displacement*, since both polygons have the same size and orientation.

**Opposite similarity**

Two triangles *ABC* and *A'B'C'* are *oppositely similar* and the sides denoted with the same letters are homologous if:

$$AB\, BC^{-1} = (\, A'B'\, B'C'^{-1}\,)^* = B'C'^{-1}\, A'B'$$

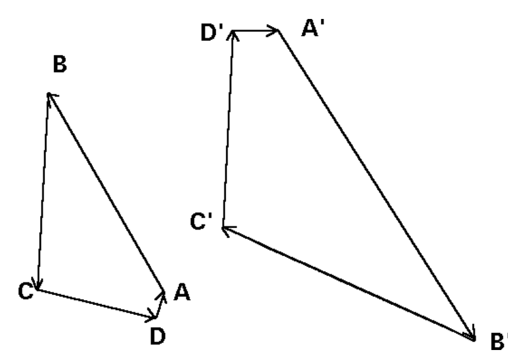


Figure 7.3

The former equality cannot be arranged in a quotient of a pair of homologous sides as we have made before. Because of this, the similarity ratio cannot be defined for the opposite similarity but only the size ratio, which is the quotient of the lengths of any two homologous sides. An opposite similarity is always the composition of a reflection in any line and a direct similarity.

$$AB\,BC^{-1} = v^{-1}\,A'B'\,B'C'^{-1}\,v \quad\Leftrightarrow\quad BC^{-1}\,v^{-1}\,B'C' = AB^{-1}\,v^{-1}\,A'B' \;\Leftrightarrow$$

$$BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = r$$

where $r$ is the ratio of a direct similarity whose argument is not defined but depends on the direction vector $v$ of the reflection axis. Notwithstanding, this expression allows to define the opposite similarity of two polygons. So two polygons $ABC...Z$ and $A'B'C'...Z'$ are oppositely similar and the sides denoted with the same letters are homologous if for any vector $v$ the following equalities are fulfilled:

$$AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = .... = ZA^{-1}\,(\,v^{-1}\,Z'A'\,v\,)$$

that is, if after a reflection one polygon is directly similar to the other. The opposite similarity is not reflexive nor transitive: if a figure is oppositely similar to another, and this is oppositely similar to a third figure, then the first and third figures are directly similar. Then there are not classes of oppositely similar figures.

An opposite similarity with $|r|=1$ is called a *reversal*, since both polygons have the same size both opposite orientations.

**The theorem of Menelaus**

For every triangle $ABC$ (figure 7.4), three points $D$, $E$ and $F$ lying respectively on the sides $BC$, $CA$ and $AB$ or their prolongations are aligned if and only if:

$$AF\,FB^{-1}\,BD\,DC^{-1}\,CE\,EA^{-1} = -1$$

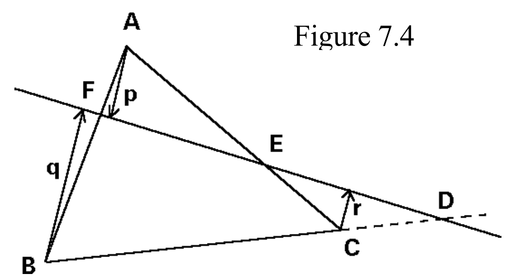


Figure 7.4

Proof $\Rightarrow$ Let us suppose that $D$, $E$ and $F$ are aligned on a crossing straight line. Let us denote by $p$, $q$ and $r$ the vectors with origin at the vertices $A$, $B$ and $C$ and going perpendicularly to the crossing line. Then every pair of right angle triangles having the hypotenuses on a common side of the triangle $ABC$ are similar so that we have:

$$BF^{-1}\,AF = q^{-1}\,p \qquad CD^{-1}\,BD = r^{-1}\,q \qquad AE^{-1}\,CE = p^{-1}\,r$$

Multiplying the three equalities one obtains:

$$AE^{-1}\,CE\,CD^{-1}\,BD\,BF^{-1}\,AF = 1$$

Taking the complex conjugate expression and changing the sign of $BF$, $CD$ and $AE$, the theorem is proved in this direction:

$$AF\,FB^{-1}\,BD\,DC^{-1}\,CE\,EA^{-1} = -1$$

Proof $\Leftarrow$ Being $F$ a point on the line $AB$, $D$ a point on the line $BC$ and $E$ a point on the line $CA$ we have:

$$F = a\,A + (1-a)\,B \qquad D = b\,B + (1-b)\,C \qquad E = c\,C + (1-c)\,A$$

with $a$, $b$ and $c$ real. Then:

$$AF = (1-a)\,AB \qquad FB = a\,AB \qquad BD = (1-b)\,BC$$

$$DC = b\,BC \qquad CE = (1-c)\,CA \qquad EA = c\,CA$$

That the product of these segments is equal to $-1$ implies that:

$$\frac{(1-a)(1-b)(1-c)}{a\,b\,c} = -1 \qquad \Rightarrow \qquad c = \frac{(1-a)(1-b)}{1-a-b}$$

The substitution of $c$ in the expression for $E$ gives:

$$E = \frac{(1-a)(1-b)}{1-a-b}C - \frac{a\,b}{1-a-b}A$$

$$= \frac{1-a}{1-a-b}\left[(1-b)C + b\,B\right] - \frac{b}{1-a-b}\left[a\,A + (1-a)B\right]$$

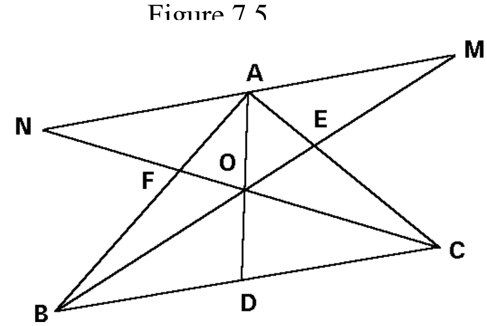
$$= \frac{1-a}{1-a-b}D - \frac{b}{1-a-b}F$$

That is, the point $E$ belongs to the line $FD$, in other words, the point $D$, $E$ and $F$ are aligned, which is the prove:

$$E = d\,D + (\,1-d\,)\,F \qquad \text{with } d = \frac{1-a}{1-a-b}$$

**The theorem of Ceva**

Figure 7.5




Given a generic triangle $ABC$, we draw three segments $AD$, $BE$ and $CF$ from each vertex to a point of the opposite side (figure 7.5). The three segments meet in a unique point if and only if:

$$BD\,DC^{-1}\,CE\,EA^{-1}\,AF\,FB^{-1} = 1$$

In order to prove the theorem, we

enlarge the segments $BE$ and $CF$ to touch the line which is parallel to the side $BC$ and passes through $A$. Let us denote the intersection points by $M$ and $N$ respectively. Then the triangle $BDO$ is directly similar to the triangle $MAO$, and the triangle $CDO$ is also directly similar to the triangle $NAO$. Therefore:

$$BD\,DO^{-1} = MA\,AO^{-1}$$

$$DO\,DC^{-1} = AO\,AN^{-1}$$

The multiplication of both equalities gives:

$$BD\,DC^{-1} = MA\,AN^{-1}$$

Analogously, the triangles $MEA$ and $CEB$ are similar as soon as $NFA$ and $BFC$. Hence:

$$CE\,EA^{-1} = BC\,AM^{-1} \qquad AF\,FB^{-1} = AN\,BC^{-1}$$

The product of the three equalities yields:

$$BD\,DC^{-1}\,CE\,EA^{-1}\,AF\,FB^{-1} = MA\,AN^{-1}\,BC\,AM^{-1}\,AN\,BC^{-1} = 1$$

The sufficiency of this condition is proved in the following way: let $O$ be the point of intersection of $BE$ and $CF$, and $D'$ the point of intersection of the line $AO$ with the side $BC$. Let us suppose that the former equality is fulfilled. Then:

$$BD'\,D'C^{-1} = BD\,DC^{-1}$$

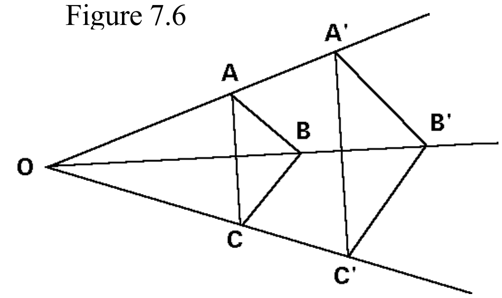
Since both $D$ and $D'$ lie on the line $BC$, it follows that $D = D'$.

### Homothety and single ratio

A *homothety with centre O and ratio k* is the geometric transformation which dilates the distance from $O$ to any point $A$ in a factor $k$:

$$OA' = OA\,k$$

Figure 7.6




$O$ is the unique invariant point of the homothety. If $k$ is real number, the homothety is said to be *simple* (figure 7.6), otherwise is called *composite*. If $k$ is a complex number, it may always be factorised in a product of the modulus and a unitary complex number:

$$k = |k|\,z \qquad \text{with } |z| = 1$$

The modulus of $k$ is the ratio of a simple homothety with centre $O$, and $z$ indicates an additional rotation with the same centre. Then, a composite homothety is equivalent to a simple homothety followed by a rotation. Direct similarities and homotheties are different names for the same transformations (exercise 7.5). Also homotheties with $|k|=1$ and displacements are equivalent.

The *single ratio of three points A*, *B*, and *C* is defined as:

$$(A\ B\ C) = AB\, AC^{-1}$$

The single ratio is a real number when the three points are aligned, and a complex number in the other case. Under a homothety, the single ratio of any three points remains invariant. Let us prove this. Because any vector is dilated and rotated by a factor *k*, we have:

$$A'B' = OB' - OA' = OB\,k - OA\,k = AB\,k$$

$$A'C' = AC\ k \qquad \Rightarrow \qquad A'C'^{-1} = k^{-1}\,AC^{-1}$$

$$(A'\ B'\ C') = AB\,k\,k^{-1}\,AC^{-1} = AB\,AC^{-1} = (A\ B\ C)$$

If the single ratio is invariant for a geometric transformation, then it transforms triangles into directly similar triangles and hence also polygons into similar polygons:

$$(A\ B\ C) = (A'\ B'\ C') \qquad \Rightarrow \quad AB\,AC^{-1} = A'B'\,A'C'^{-1} \quad \Rightarrow AC^{-1}\,A'C' = AB^{-1}\,A'B'$$

Therefore, the homothety always transforms triangles into directly similar triangles as shown in the figure 7.6. It follows immediately that the homothety preserves the angles between lines. It is the simplest case of *conformal* transformations, the geometric transformations which preserve the angles between lines. Since a line may also be transformed into a curve, and for the general case a curve into a curve, the conformal transformations are those which preserve the angle between any pair of curves, that is, the angle between the tangent lines to both curves at the intersection point. A transformation is *directly* or *oppositely conformal* whether it preserves or changes the sense of the angle between two curves (figure 7.7). This condition is equivalent to the conservation of the single ratio of any three points at the limit of accumulation:

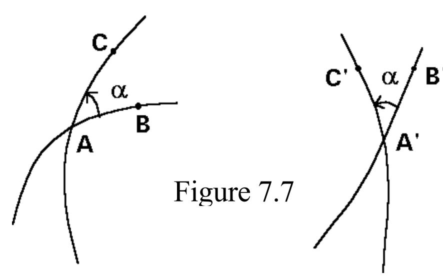


Figure 7.7

$$\lim_{B,C\to A} (A\,B\,C) = \lim_{B',C'\to A'} (A'\,B'\,C') \qquad \text{directly conformal}$$

$$\lim_{B,C\to A} (A\,B\,C) = \lim_{B',C'\to A'} (A'\,B'\,C')^* \qquad \text{oppositely conformal}$$

**Exercises**

7.1 Let $T$ be the triangle with vertices (0, 0), (2, 0), (0, 1) and $T'$ that with vertices (2,0), (5, 1) and (4, 2). Find which vertices are homologous and calculate the similarity ratio. Which is the size ratio? Which is the angle of rotation of the triangle $T'$ with respect to $T$?

7.2 The altitude perpendicular to the hypotenuse divides a right triangle in two smaller right triangles. Show that they are similar and deduce the Pythagorean theorem.

7.3 Every triangle $ABC$ with not vanishing area has a circumscribed circle. The line tangent to this circle at the point $B$ cuts the line $AC$ at the point $M$. Prove that:

$$MA\, MC^{-1} = AB^2\, BC^{-2}$$

7.4 Let $ABC$ be an equilateral triangle inscribed inside a circle. If $P$ is any point on the arc $BC$, show that $|PA| = |PB| + |PC|$.

7.5 Given two directly similar triangles $ABC$ and $A'B'C'$, show that the centre $O$ of the homothety that transform one triangle into another is equal to:

$$O = A - AA'\,(1 - AB^{-1}\, A'B')^{-1}$$

7.6 Draw any line passing through a fixed point $P$ which cuts a given circle. Let $Q$ and $Q'$ be the intersection points of the line and the circle. Show that the product $PQ\, PQ'$ is constant for any line belonging to the pencil of lines of $P$.

7.7 Let a triangle have sides $a$, $b$ and $c$. The bisector of the angle formed by the sides $a$ and $b$ divides the side $c$ in two parts $m$ and $n$. If $m$ is adjacent to $a$ and $n$ to $b$ respectively, prove that the following proportion is fulfilled:

$$\frac{|m|}{|a|} = \frac{|n|}{|b|}$$

## 8. PROPERTIES OF TRIANGLES

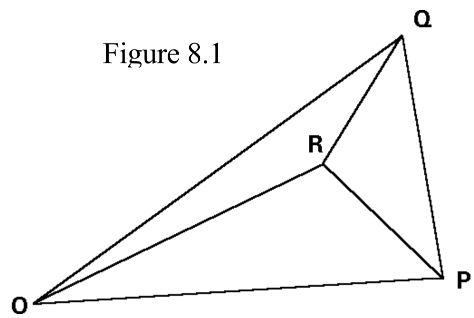
### Area of a triangle

Since the area of a parallelogram is obtained as the outer product of two consecutive sides (taken as vectors, of course), the area of a triangle is the half of the outer product of any two of its three sides:

$$a_{PQR} = \frac{1}{2}\, PQ \wedge PR = \frac{1}{2}\, QR \wedge QP = \frac{1}{2}\, RP \wedge RQ$$

If the vertices $P$, $Q$ and $R$ are counterclockwise oriented, the area is a positive imaginary number. Otherwise, the area is a negative imaginary number. Note that the fundamental concept in geometry is the *oriented area*[1]. The modulus of the area may be useful in the current life but is insufficient for geometry. From now on I shall only regard oriented areas.

Figure 8.1




Writing the segments of the former equation as differences of points we arrive to:

$$a_{PQR} = \frac{1}{2}\,(P \wedge Q + Q \wedge R + R \wedge P\,)$$

which is a symmetric expression under cyclic permutation of the vertices. The position vector $P$ goes from an arbitrary origin of coordinates to the point $P$. Then, $P \wedge Q$ is the double of the area of the triangle $OPQ$. Analogously $Q \wedge R$ is the double of the area of the triangle $OQR$ and $R \wedge P$ is the double of the area of the triangle $ORP$. Therefore the former expression is equal to (figure 8.1):

$$a_{PQR} = a_{OPQ} + a_{OQR} + a_{ORP}$$

For the arrangement of points shown in the figure 8.1 the area of $OPQ$ is positive, and the areas of $OQR$ and $ORP$ are negative, that is, the areas of $ORQ$ and $OPR$ are positive. Therefore, one would intuitively write, taking all the areas positive, that:

$$a_{PQR} = a_{OPQ} - a_{ORQ} - a_{OPR}$$

When one considers oriented areas, the equalities are wholly general and independent of the arrangement of the points[2].

---

[1] The integral of a function is also the oriented area of the region enclosed by the curve and the $X$-axis.

[2] About this topic, see A. M. Lopshitz, *Cálculo de las áreas de figuras orientadas*, Rubiños-1860 (1994).

**Medians and centroid**

The *medians* are the segments going from each vertex to the midpoint of the opposite side. Let us prove that the three medians meet in a unique point $G$ called the *centroid* (figure 8.2). Since $G$ is a point on the median passing through $P$ and $(Q + R)/2$:

$$G = k\,P + (1-k)\frac{Q+R}{2} \qquad k \text{ real}$$

$G$ lies also on the median passing through $Q$ and $(P + R) / 2$:

$$G = m\,Q + (1-m)\frac{P+R}{2} \qquad m \text{ real}$$

Equating both expressions we find:

$$k\,P + (1-k)\frac{Q+R}{2} = m\,Q + (1-m)\frac{P+R}{2}$$

$$P\left(k - \frac{1}{2} + \frac{m}{2}\right) + Q\left(\frac{1}{2} - \frac{k}{2} - m\right) + R\left(-\frac{k}{2} + \frac{m}{2}\right) = 0$$

A linear combination of independent points can vanish only if every coefficients are null, a condition which leads to the following system of equations:

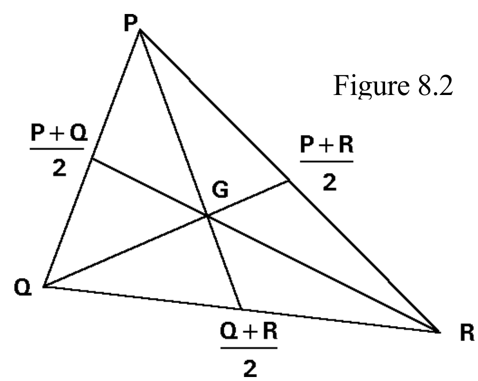


Figure 8.2

$$\begin{cases} k - \dfrac{1}{2} + \dfrac{m}{2} = 0 \\ \dfrac{1}{2} - \dfrac{k}{2} - m = 0 \\ -\dfrac{k}{2} + \dfrac{m}{2} = 0 \end{cases}$$

The solution of this system of equations is $k = 1/3$ and $m = 1/3$, indicating that the intersection of both medians are located at 1/3 distance from the midpoints. The substitution into the expression of $G$ gives:

$$G = \frac{P+Q+R}{3}$$

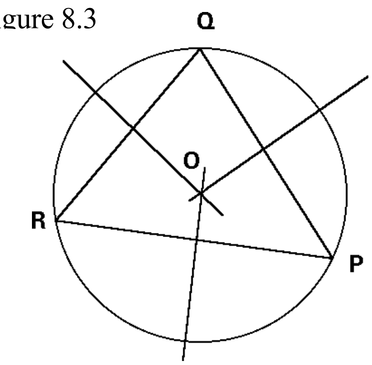
This expression for the centroid is symmetric under permutation of the vertices. Therefore the three medians meet at the same point $G$, the centroid[3].

[3] The medians are a special case of *cevian lines* (lines passing through a vertex and not coinciding with the sides) and this statement is also proved by means of Ceva's theorem.

**Perpendicular bisectors and circumcentre**

The three perpendicular bisectors of the sides of a triangle meet in a unique point called the *circumcentre,* the centre of the circumscribed circle. Every point on the perpendicular bisector of $PQ$ is equidistant from $P$ and $Q$. Analogously every point on the perpendicular bisector of $PR$ is equidistant from $P$ and $R$. The intersection $O$ of both perpendicular bisectors is simultaneously equidistant from $P$, $Q$ and $R$. Therefore $O$ also belongs to the perpendicular bisector of $QR$ and the three bisectors meet at a unique point. Since $O$ is equally distant from the three vertices, it is the centre of the circumscribed circle. Let us use this condition in order to calculate the equation of the circumcentre:

Figure 8.3




$$OP^2 = OQ^2 = OR^2 = d^2$$

where $d$ is the radius of the circumscribed circle. Using the position vectors of each point we have:

$$(P - O)^2 = (Q - O)^2 = (R - O)^2$$

The first equality yields:

$$P^2 - 2\, P \cdot O + O^2 = Q^2 - 2\, Q \cdot O + O^2$$

By simplifying and arranging the terms containing $O$ at the left hand side, we have:

$$2\,(\,Q - P\,) \cdot O = Q^2 - P^2$$

$$2\, PQ \cdot O = Q^2 - P^2$$

From the second equality we find an analogous result:

$$2\, QR \cdot O = R^2 - Q^2$$

Now we introduce the geometric product instead of the inner product in these equations:

$$PQ\ \ O + O\, PQ = Q^2 - P^2$$

$$QR\, O + O\, QR = R^2 - Q^2$$

By subtraction of the second equation multiplied on the right by $PQ$ minus the first equation multiplied on the left by $QR$, we obtain:

$$PQ\ \ QR\ \ O - O\, PQ\ \ QR = PQ\ \ R^2 - PQ\ \ Q^2 - Q^2\ \ QR + P^2\ \ QR$$

By using the permutative property on the left hand side and simplifying the right hand side, we have:

$$PQ\ QR\ O - QR\ PQ\ O = P^2\ QR + Q^2\ RP + R^2\ PQ$$

$$2\ (PQ \wedge QR)\ O = P^2\ QR + Q^2\ RP + R^2\ PQ$$

Finally, the multiplication by the inverse of the outer product on the left gives:

$$O = (2\ PQ \wedge QR)^{-1}\ (P^2\ QR + Q^2\ RP + R^2\ PQ)$$

$$= -\ (P^2\ QR + Q^2\ RP + R^2\ PQ)\ (2\ PQ \wedge QR)^{-1}$$

a formula able to calculate the coordinates of the circumcentre. For example, let us calculate the centre of the circle passing through the points:

$$P = (2, 2) \qquad Q = (3, 1) \qquad R = (4, -2)$$

$$P^2 = 8 \qquad Q^2 = 10 \qquad R^2 = 20$$

$$QR = R - Q = e_1 - 3\ e_2 \qquad RP = P - R = -\ 2\ e_1 + 4\ e_2 \qquad PQ = Q - P = e_1 - e_2$$

$$2\ PQ \wedge QR = -\ 4\ e_{12}$$

$$O = -\ (8\ (e_1 - 3\ e_2) + 10\ (-2\ e_1 + 4\ e_2) + 20\ (e_1 - e_2))\frac{e_{12}}{4}$$

$$= -\ e_1 - 2\ e_2 = (-1, -2)$$

In order to deduce the radius of the circle, we take the vector $OP$:

$$OP = P - O = P + (P^2\ QR + Q^2\ RP + R^2\ PQ)\ (2\ PQ \wedge QR)^{-1}$$

and extract the inverse of the area as a common factor:

$$OP = (2\ P\ \ PQ \wedge QR + P^2\ QR + Q^2\ RP + R^2\ PQ)\ (2\ PQ \wedge QR)^{-1} =$$

$$= [\ 2\ P\ (P \wedge Q + Q \wedge R + R \wedge P) + P^2\ QR + Q^2\ RP + R^2\ PQ\ ]\ (2\ PQ \wedge QR)^{-1} =$$

$$= [\ P\ (P\ Q - Q\ P + Q\ R - R\ Q + R\ P - P\ R) + P^2\ (R - Q) + Q^2\ (P - R) +$$

$$+ R^2\ (Q - P)\ ]\ \ (2\ PQ \wedge QR)^{-1}$$

The simplification gives:

$$OP = (P\ Q\ R - P\ R\ Q + P\ R\ P - P\ Q\ P + Q^2\ P - Q^2\ R + R^2\ Q - R^2\ P)\ (2\ PQ \wedge QR)^{-1}$$

$$= -\ (Q - P)\ (R - Q)\ (P - R)\ (2\ PQ \wedge QR)^{-1} = -\ PQ\ \ QR\ \ RP\ (2\ PQ \wedge QR)^{-1}$$

Analogously:

$$OQ = -\,QR\ \ RP\ \ PQ\,(\,2\,PQ \wedge QR\,)^{-1} \qquad OR = -\,RP\ \ PQ\ \ QR\,(\,2\,PQ \wedge QR\,)^{-1}$$

The radius of the circumscribed circle is the length of any of these vectors:

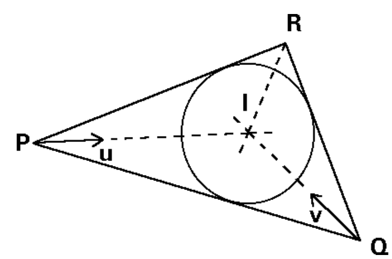
$$|\,OP\,| = \frac{|\,PQ\,|\,|\,QR\,|\,|\,RP\,|}{2\,|\,PQ \wedge QR\,|} = \frac{|\,PQ\,|}{2\sin QRP} = \frac{|\,QR\,|}{2\sin RPQ} = \frac{|\,RP\,|}{2\sin PQR}$$

where we find the law of sines.

### Angle bisectors and incentre

The three bisector lines of the angles of a triangle meet in a unique point called the *incentre*. Every point on the bisector of the angle with vertex $P$ is equidistant from the sides $PQ$ and $PR$ (figure 8.4). Also every point on the angle bisector of $Q$ is equidistant from the sides $QR$ and $QP$. Hence its intersection $I$ is simultaneously equidistant from the three sides, that is, $I$ is unique and is the centre of the circle inscribed into the triangle.

Figure 8.4




In order to calculate the equation of the angle bisector passing through $P$, we take the sum of the unitary vectors of both adjacent sides:

$$u = \frac{PQ}{|\,PQ\,|} + \frac{PR}{|\,PR\,|} \qquad\qquad v = \frac{QP}{|\,QP\,|} + \frac{QR}{|\,QR\,|}$$

The incentre $I$ is the intersection of the angle bisector passing through $P$, whose direction vector is $u$, and that passing through $Q$, with direction vector $v$:

$$I = P + k\,u = Q + m\,v \qquad\qquad k,\ m \text{ real}$$

Arranging terms we find $PQ$ as a linear combination of $u$ and $v$:

$$k\,u - m\,v = Q - P = PQ$$

The coefficient $k$ is:

$$k = \frac{PQ \wedge v}{u \wedge v} = \frac{|\,PQ\,|\,|\,RP\,|\,PQ \wedge QR}{PQ \wedge QR\,|\,RP\,| + QR \wedge RP\,|\,PQ\,| + RP \wedge PQ\,|\,QR\,|}$$

Since all outer products are equal because they are the double of triangle area, this expression is simplified:

$$k = \frac{|PQ|\,|RP|}{|RP| + |PQ| + |QR|}$$

Then, the centre of the circumscribed circle is:

$$I = P + k\,u = P + \frac{|PQ|\,|RP|}{|PQ| + |QR| + |RP|}\left(\frac{PQ}{|PQ|} + \frac{PR}{|PR|}\right)$$

By taking common denominator and simplifying, we arrive at:

$$I = \frac{P|QR| + Q|RP| + R|PQ|}{|QR| + |RP| + |PQ|}$$

For example, let us calculate the centre of the circle inscribed inside the triangle with vertices:

$P = (0, 0) \qquad Q = (0, 3) \qquad R = (4, 0)$

$|PQ| = 3 \qquad |QR| = 5 \qquad |RP| = 4$

$$I = \frac{5(0,0) + 4(0,3) + 3(4,0)}{5 + 4 + 3} = \frac{(12,12)}{12} = (1,1)$$

In order to find the radius, firstly we must obtain the segment *IP*:

$$IP = \frac{QP|RP| + RP|PQ|}{|QR| + |RP| + |PQ|}$$

The radius of the inscribed circle is the distance from *I* to the side *PQ*:

$$d(I, PQ) = \frac{|IP \wedge PQ|}{|PQ|} = \frac{|RP \wedge PQ|}{|PQ| + |QR| + |RP|}$$

whence the ratio of radius follows:

$$\frac{\text{radius of circumscribed circle}}{\text{radius of inscribed circle}} = \frac{1}{2}\,\frac{|PQ|\,|QR|\,|RP|}{|PQ| + |QR| + |RP|}$$

**Altitudes and orthocentre**

The *altitude* of a side is the segment perpendicular to this side (also called *base*) which passes through the opposite vertex. The three altitudes of a triangle intersect on a unique point called the *orthocentre*. Let us prove this statement calculating the intersection $H$ of two altitudes. Since $H$ belongs to the altitude passing through the vertex $P$ and perpendicular to the base $QR$ (figure 8.5), its equation is:

$$H = P + z\, QR \qquad z \text{ imaginary}$$

because the product by an imaginary number turns the vector $QR$ over $\pi/2$, that is, $z\, PQ$ has the direction of the altitude. $H$ also belongs to the altitude passing through $Q$ and perpendicular to the base $RP$. Then its equation is:

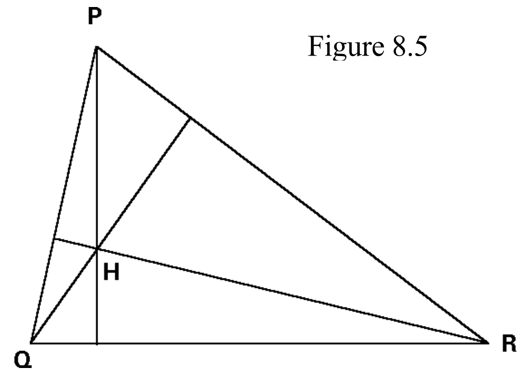


Figure 8.5

$$H = Q + t\, RP \qquad t \text{ imaginary}$$

By equating both expressions:

$$P + z\, QR = Q + t\, RP$$

we arrive at a vector written as a linear combination of two vectors but with imaginary coefficients:

$$z\, QR - t\, RP = PQ$$

In this equation we must resolve $PQ$ into components with directions perpendicular to the vectors $QR$ and $RP$. The algebraic resolution follows the same way as for the case of real linear combination. Let us multiply on the right by $RP$:

$$z\, QR\, RP - t\, RP^2 = PQ\, RP$$

and on the left:

$$RP\, z\, QR - RP\, t\, RP = RP\, PQ$$

The imaginary numbers anticommute with vectors:

$$- z\, RP\, QR + t\, RP^2 = RP\, PQ$$

By adding both equalities we arrive at:

$$z\, (\, QR\, RP - RP\, QR\,) = PQ\, RP + RP\, PQ$$

$$z = \frac{PQ\, RP + RP\, PQ}{QR\, RP - RP\, QR} = \frac{PQ \cdot RP}{QR \wedge RP}$$

Analogously one finds $t$:

$$t = \frac{PQ \cdot QR}{QR \wedge RP}$$

Both expressions differ from the coefficients of real linear combination in the fact that the numerator is an inner product. The substitution in any of the first equations gives:

$$H = P + z\ QR = P + (\ PQ\ RP + RP\ PQ\ )\ (\ QR\ RP - RP\ QR\ )^{-1}\ QR =$$

The vector $QR$ anticommutes with the outer product, an imaginary number:

$$H = P - (\ PQ\ RP + RP\ PQ\ )\ QR\ (\ QR\ RP - RP\ QR\ )^{-1} =$$

Extracting the area as common factor, we obtain:

$$H = [\ P\ (\ QR\ RP - RP\ QR\ ) - (\ PQ\ RP + RP\ PQ\ )\ QR\ ]\ (\ QR\ RP - RP\ QR\ )^{-1}$$

By using the fact that $PQ = Q - P$, etc., we arrive at:

$$H = (\ P\ QR\ RP - Q\ RP\ QR - P\ PQ\ QR + R\ PQ\ QR\ )\ (\ QR\ RP - RP\ QR\ )^{-1}$$

$$= (\ P^2\ QR + Q^2\ RP + R^2\ PQ + P\ QR\ P + Q\ RP\ Q + R\ PQ\ R\ )\ (\ QR\ RP - RP\ QR\ )^{-1}$$

$$= (\ P\ \ P \cdot QR + Q\ \ Q \cdot RP + R\ \ R \cdot PQ\ )\ (\ QR \wedge RP\ )^{-1}$$

This formula is invariant under cyclic permutation of the vertices. Therefore all the altitudes intersect on a unique point, the orthocentre.

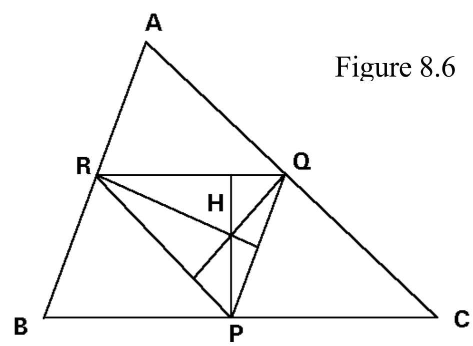



Figure 8.6

The equation of the orthocentre resembles that of the circumcentre. In order to see the relationship between both, let us draw a line passing through $P$ and parallel to the opposite side $QR$, another one passing through $Q$ and parallel to $RP$, and a third line passing through $R$ and parallel to $PQ$ (figure 8.6). Let $A$ be the intersection of the line passing through $Q$ and that passing through $R$, $B$ be the intersection of the lines passing through $R$ and $P$ respectively, and $C$ be the intersection of the lines passing through $P$ and $Q$ respectively. The triangle $ABC$ is directly similar to the triangle $PQR$ with ratio $-2$:

$$PQ = -\frac{1}{2}AB \qquad QR = -\frac{1}{2}BC \qquad RP = -\frac{1}{2}CA$$

and $P$, $Q$ and $R$ are the midpoints of the sides of the triangle $ABC$:

$$P = \frac{B+C}{2} \qquad Q = \frac{C+A}{2} \qquad R = \frac{A+B}{2}$$

Therefore the altitudes of the triangle $PQR$ are the perpendicular bisectors of the sides of the triangle $ABC$, and the orthocentre of the triangle $PQR$ is the circumcentre of

$ABC$. In order to prove with algebra this obvious geometric fact, we must only deduce from the former relations the following equalities and substitute them into the orthocentre equation:

$$P\ P\cdot QR = \frac{B^3 - B\ C^2 + C\ B^2 - C^3}{8}$$

$$Q\ Q\cdot RP = \frac{C^3 - C\ A^2 + A\ C^2 - A^3}{8}$$

$$R\ R\cdot PQ = \frac{A^3 - A\ B^2 + B\ A^2 - B^3}{8}$$

By adding the three terms, the cubic powers vanish. On the other hand, the area of the triangle $PQR$ is four times smaller than the area of the triangle $ABC$:

$$QR\wedge RP = \frac{BC\wedge CA}{4}$$

$$H = (\,-B\,C^2 + C\,B^2 - C\,A^2 + A\,C^2 - A\,B^2 + B\,A^2\,)\,(\,2\,BC\wedge CA\,)^{-1} =$$

$$= -\,(\,A^2\,BC + B^2\,CA + C^2\,AB\,)\,(\,2\,BC\wedge CA\,)^{-1}$$

This is just the equation of the circumcentre of the triangle $ABC$.

### Euler's line

The centroid $G$, the circumcentre $O$ and the orthocentre $H$ of a triangle are always aligned on the *Euler's line*. To prove this fact, observe that the circumcentre and orthocentre equations have the triangle area as a "denominator", while the centroid equation does not, but we can introduce it:

$$G = \frac{P+Q+R}{3} = (\,P+Q+R\,)\,QR\wedge RP\,(\,3\,QR\wedge RP\,)^{-1}$$

Introducing the equality:

$$QR\wedge RP = P\wedge Q + Q\wedge R + R\wedge P = \frac{P\ Q - Q\ P + Q\ R - R\ Q + R\ P - P\ R}{2}$$

the centroid becomes:

$$G = (\,P+Q+R\,)\,(\,P\,Q - Q\,P + Q\,R - R\,Q + R\,P - P\,R\,)\,(\,6\,QR\wedge RP\,)^{-1}$$

$$= (\,P\,P\,Q - P\,Q\,P + P\,R\,P - P\,P\,R + Q\,P\,Q - Q\,Q\,P + Q\,Q\,R - Q\,R\,Q +$$

$$R\,Q\,R - R\,R\,Q + R\,R\,P - R\,P\,R\,)\,(\,6\,QR\wedge RP\,)^{-1}$$

$$= (\,-P^2\,QR + P\,QR\,P - Q^2\,RP + Q\,RP\,Q - R^2\,PQ + R\,PQ\,R\,)\,(\,6\,QR\wedge RP\,)^{-1}$$

from where the relation between the three points $G$, $H$ and $O$ follows immediately:

$$G = \frac{H + 2\,O}{3}$$

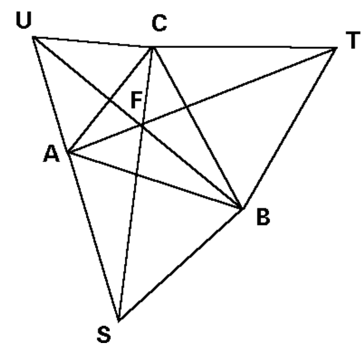
Hence the centroid is located between the orthocentre and the circumcentre, and its distance from the orthocentre is double of its distance from the circumcentre.

**The Fermat's theorem**

The geometric algebra allows to prove through a very easy and intuitive way the Fermat's theorem.

Over every side of a triangle $ABC$ we draw an equilateral triangle (figure 8.7). Let $T$, $U$ and $S$ be the vertices of the equilateral triangles respectively opposite to $A$, $B$ and $C$. Then the segments $AT$, $BU$ and $CS$ have the same length, form angles of $2\pi/3$ and intersect on a unique point $F$, called the *Fermat's point*. Moreover, the addition of the three distances from any point $P$ to each vertex is minimal when $P$ is the Fermat's point, provided that any of the interior angles of the triangle $ABC$ be higher than $2\pi/3$.

Figure 8.7




Firstly, we must demonstrate that $BU$ is obtained from $AT$ by means of a rotation of $2\pi/3$, which will be represented by the complex number $t$:

$$t = \cos\frac{2\pi}{3} + e_{12}\,\sin\frac{2\pi}{3}$$

$$AT\,t = (AC + CT\,)\,t = AC\,t + CT\,t$$

By construction, the vector $AC$ turned over $2\pi/3$ is the vector $CU$, and $CT$ turned over $2\pi/3$ is $BC$, so:

$$AT\,t = CU + BC = BU$$

Analogously, one finds $CB = BU\,t$ and $AT = CS\,t$ . That is, the vectors $CS$, $BU$ and $AT$ have the same length and each one is obtained from each other by successive rotations of $2\pi/3$.

Let us see that the sum of the distances from $P$ to the three vertices $A$, $B$ and $C$ is minimal when $P$ is the Fermat's point. We must prove firstly that the vectorial sum of $PA$ turned $4\pi/3$, $PB$ turned $2\pi/3$ and $PC$ is constant independently of the point $P$. (figure 8.8). That is, for any two points $P$ and $P'$ it is always true that:

$$PA\, t^2 + PB\, t + PC = P'A\, t^2 + P'B\, t + P'C$$

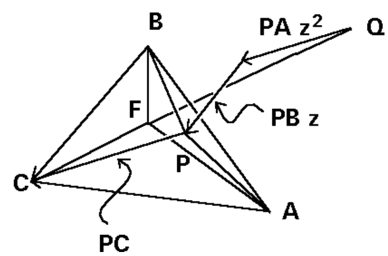
fact which is proved by arranging all the terms in one side of the equation:

$$PP'\,(\, t^2 + t + 1\,) = 0$$

This product is always zero since $t^2 + t + 1 = 0$. Hence, there is a unique point $Q$ such that:

$$PA\, t^2 + PB\, t + PC = QC$$

Figure 8.8




For any point $P$, the three segments form a broken line as that shown in figure 8.8. Therefore, by the triangular inequality we have:

$$|PA| + |PB| + |PC| \geq |QC|$$

When $P$ is the Fermat's point $F$, these segments form a straight line. Then, the addition of the distances from $F$ to the three vertices is minimal provided that no angle of the triangle be higher than $2\pi/3$:

$$|FA| + |FB| + |FC| = |QC| \leq |PA| + |PB| + |PC|$$

In the other case, someone of the vectors $FA\, t^2$, $FB\, t$ or $FC$ has a sense opposite to the others, so that its length is subtracted from the others and their sum is not minimal.

### Exercises

8.1 The Napoleon's theorem. Over each side of a generic triangle draw an equilateral triangle. Prove that the centres of the three equilateral triangles also form an equilateral triangle.

8.2 Prove the Leibniz's theorem. Let $P$ be any point on the plane and $G$ the centroid of a triangle $ABC$. Then the following equality holds:

$$3\, PG^2 = PA^2 + PB^2 + PC^2 - \frac{1}{3}(\, AB^2 + BC^2 + CA^2\,)$$

8.3 Let $A$, $B$ and $C$ be three given points on the plane. Then any point $G$ on the plane can be expressed as a linear combination of these three points ($G$ is also considered as the centre of masses located at $A$, $B$ and $C$ with weights $a$, $b$ and $c$ [4]).

[4] See August Ferdinand Möbius, *Der Barycentrische Calcul*, Leipzig, 1827, p. 17 (facsimile edition of Georg Olms Verlag, Hildesheim,1976).

$G = a\,A + b\,B + c\,C \qquad\qquad$ with $\;a + b + c = 1$

Prove that:

a) $a$, $b$, $c$ are the fractions of the area of the triangle $ABC$ occupied by the triangles $GBC$, $GCA$ and $GAB$ respectively.

b) the geometric locus of the points $P$ on the plane such that:

$$a\,PA^2 + b\,PB^2 + c\,PC^2 = k$$

is a circle with centre $G$ (Apollonius' lost theorem).

8.4 Over every side of a convex quadrilateral $ABCD$, we draw the equilateral triangles $ABP$, $BCQ$, $CDR$ and $DAS$. Prove that:

a) If $|AC| = |BD|$ then $PR$ is perpendicular to $QS$.

b) If $|PR| = |QS|$ then $AC$ is perpendicular to $BD$.

8.5 Show that the length of a median of a triangle $ABC$ passing through $A$ can be calculated from the sides as:

$$m_A^2 = \frac{AB^2 + AC^2}{2} - \frac{BC^2}{4}$$

8.6 In a triangle $ABC$ we draw the bisectors of the angles $A$ and $B$. Through the vertex $C$ we draw the lines parallel to each angle bisector. Let us denote the intersections of each parallel line with the other bisector by $D$ and $E$. Prove that if the line $DE$ is parallel to the side $AB$ then the triangle $ABC$ is isosceles.

8.7 The bisection of a triangle (proposed by Cristóbal Sánchez Rubio). Let $P$ and $Q$ be two points on different sides of a triangle such that the segment $PQ$ divides the triangle in two parts with the same area. Calculate the segment $PQ$ in the following cases:

a) $PQ$ is perpendicular to a given direction.

b) $PQ$ has minimum length.

c) $PQ$ passes through a given point inside the triangle.

## 9. CIRCLES

### Algebraic and Cartesian equations

A circle with radius $r$ and centre $F$ is the locus of the points located at a distance $r$ from $F$, so its equation is:

$$d(F, P) = |FP| = r \quad \Leftrightarrow \quad FP^2 = r^2$$

which can be written as:

$$(P - F)^2 = r^2 \quad \Rightarrow \quad P^2 - 2\, P \cdot F + F^2 - r^2 = 0$$

By introducing the coordinates of $P = (x, y)$ and $F = (a, b)$ one obtains the analytic equation of the circle:

$$x^2 + y^2 - 2\, a\, x - 2\, b\, y + a^2 + b^2 - r^2 = 0$$

For example, the circle whose equation is:

$$x^2 + y^2 + 4\, x - 6\, y + 9 = 0$$

has the centre $F = (-2, 3)$ and radius 2.

There always exists a unique circle passing through any three non aligned points. In order to obtain this circle, one may substitute the coordinates of the points on the Cartesian equation of the circle, arriving at a system of three linear equations of three unknown quantities $a$, $b$ and $c$:

$$\begin{cases} x_1^{\,2} + y_1^{\,2} + a\, x_1 + b\, y_1 + c = 0 \\ x_2^{\,2} + y_2^{\,2} + a\, x_2 + b\, y_2 + c = 0 \\ x_3^{\,2} + y_3^{\,2} + a\, x_3 + b\, y_3 + c = 0 \end{cases}$$

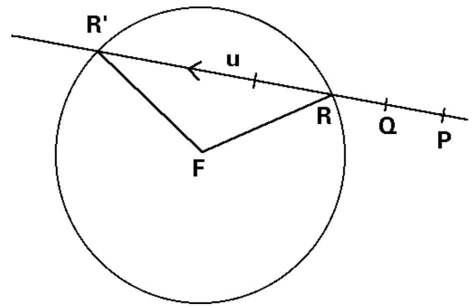
This is the classic way to find the equation of the circle, and from here the centre of the circle.

However, a more direct way is the calculation of the circumcentre of the triangle whose vertices are the three given points (see the previous chapter). Then the radius is the distance from the centre to any vertex.

### Intersections of a line with a circle

The coordinates of the intersection of a line with a circle are usually calculated by substitution of the line equation into the circle equation, which leads to a second degree equation. Let us now follow an algebraic way. Let $F$ be the centre of the circle with radius $r$, and $R$ and $R'$ the intersections of the line passing through two given points $P$ and $Q$ (figure 9.1):

Figure 9.1




$$\begin{cases} FR^2 = r^2 \\ R = P + k\,PQ \end{cases}$$

From the first equality we obtain:

$$(R - F)^2 = R^2 - 2\,R \cdot F + F^2 = r^2$$

The substitution of $R$ given by the second equality yields:

$$(P + k\,PQ)^2 - 2\,(P + k\,PQ) \cdot F + F^2 = r^2$$

$$P^2 + 2\,k\,P \cdot PQ + k^2\,PQ^2 - 2\,P \cdot F - 2\,k\,PQ \cdot F + F^2 - r^2 = 0$$

By arranging the powers of $k$ one obtains:

$$k^2\,PQ^2 + 2\,k\,(P \cdot PQ - F \cdot PQ) + P^2 - 2\,F \cdot P + F^2 - r^2 = 0$$

Taking into account that $PF = F - P$ one finds:

$$k^2\,PQ^2 - 2\,k\,PF \cdot PQ + PF^2 - r^2 = 0$$

This equation has solution whenever the discriminant be positive:

$$4\,(PF \cdot PQ)^2 - 4\,PQ^2\,(PF^2 - r^2) \geq 0$$

Introducing the identity $(PF \cdot PQ)^2 - (PF \wedge PQ)^2 = PQ^2\,PF^2$ the discriminant becomes:

$$4\,(PF \wedge PQ)^2 + 4\,PQ^2\,r^2 \geq 0$$

The solution of the second degree equation for $k$ is:

$$k = PF \cdot PQ^{-1} \pm \sqrt{r^2\,PQ^{-2} + \left(PF \wedge PQ^{-1}\right)^2}$$

When $r = |PF \wedge PQ| / |PQ|$, the line is tangent to the circle. In this case the height of the parallelogram formed by the vectors $PF$ and $PQ$ is equal to the radius of the circle.

If $u$ denote the unitary direction vector of the line $PQ$ (figure 9.1):

$$u = \frac{PQ}{|PQ|}$$

then both intersection points are written as:

$$R = P + kPQ = P + u\left( PF \cdot u - \sqrt{r^2 + (PF \wedge u)^2}\right)$$

$$R' = P + k' PQ = P + u\left( PF \cdot u + \sqrt{r^2 + (PF \wedge u)^2}\right)$$

**Power of a point with respect to a circle**

Both intersection points $R$ and $R'$ have the following property: the product of the vectors going from a given point $P$ to the intersections $R$ and $R'$ of any line passing through $P$ with a given circle is constant:

$$PR\ PR' = (PF \cdot u)^2 - r^2 - (PF \wedge u)^2$$

$$= PF^2 - r^2 = PT^2$$

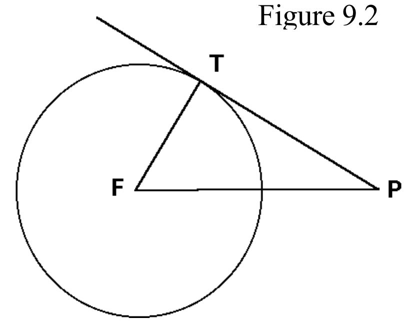


Figure 9.2

where $T$ is the contact point of the tangent line passing through $P$ (figure 9.2). The product $PR\ PR'$, which is independent of the direction $u$ of the line and only depends on the point $P$, is called *the power of the point P with respect to the given circle*. The power of a point may be calculated through the substitution of its coordinates into the circle equation. If its centre is $F = (a, b)$ and $P = (x, y)$, then:

$$FP^2 - r^2 = (x - a)^2 + (y - b)^2 - r^2 = x^2 + y^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 - r^2$$

The power is equal to zero when $P$ belongs to the circle, positive when $P$ lies outside the circle, and negative when $P$ lies inside the circle.

**Polar equation**

We wish to describe the distance from a point $P$ to the points $R$ and $R'$ on the circle as a function of the angle $\alpha$ between the line $PR$ and the diameter (figure 9.3). The vectors $PR$ and $PR'$ are:

$$PR = u\left( PF \cdot u - \sqrt{r^2 - |\,PF \wedge u\,|^2}\right)$$

$$PR' = u\left( PF \cdot u + \sqrt{r^2 + |\,PF \wedge u\,|^2}\right)$$

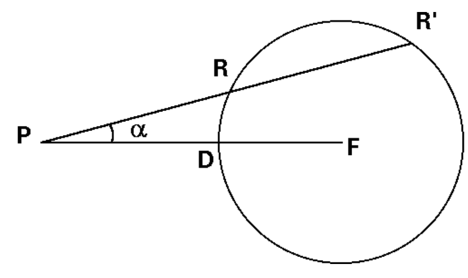


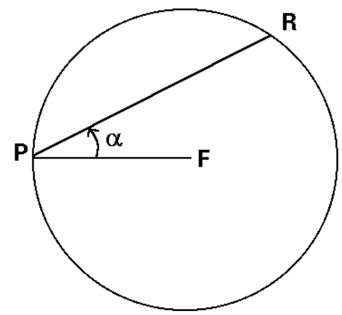
Figure 9.3

We arrive at the *polar equation* by taking the modulus of $PR$ and $PR'$, which is the distance to the circle:

$$|PR| = |PF|\cos\alpha - \sqrt{r^2 - PF^2\sin^2\alpha}$$

$$|PR'| = |PF|\cos\alpha + \sqrt{r^2 - PF^2\sin^2\alpha}$$

Figure 9.4




If $P$ is a point on the circle then $|PF| = r$ (figure 9.4) and the equation is simplified:

$$|PR| = 2\,r\cos\alpha$$

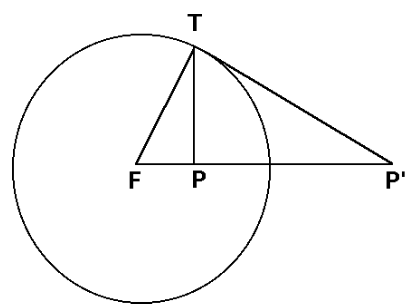
**Inversion with respect to a circle**

The point $P'$ is the *inverse point* of $P$ with respect to a circle of centre $F$ and radius $r$ if the vector $FP'$ is the inverse vector of $FP$ with radius $k$:

$$FP' = k^2\, FP^{-1} \qquad \Leftrightarrow \qquad FP\ FP' = k^2$$

Obviously, the product of the vectors $FP$ and $FP'$ is only real and positive when the points $P$ and $P'$ are aligned with the centre $F$ and are located at the same side of $F$. The circle of centre $F$ and radius $k$ is called the *inversion circle* because its points remain invariant under this inversion. The inversion transforms points located inside the inversion circle into outside points and reciprocally.

Figure 9.5




To obtain geometrically the inverse of an inside point $P$, draw firstly (figure 9.5) the diameter passing through $P$; after this draw the perpendicular to this diameter from $P$ which will cut the circle in the point $T$; finally draw the tangent with contact point $T$. The intersection $P'$ of this tangent with the prolongation of the diameter is the inverse point of $P$. Note that the right triangles $FPT$ and $FTP'$ are oppositely similar:

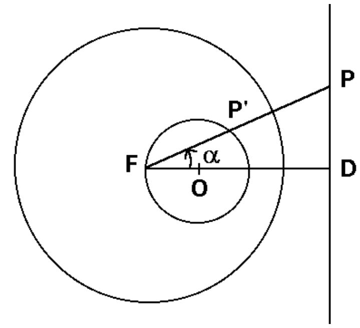
$$FP\,FT^{-1} = FP'^{-1}\,FT \;\Rightarrow\; FP'\,FP = FT^2 = k^2$$

To obtain the inverse of an outside point, make the same construction but in the opposite sense. Draw the tangent to the circle passing through the point ($P'$ in the figure 9.5) and then draw the perpendicular to the diameter $FP'$ passing through the contact point $T$ of the tangent. The intersection $P$ of the perpendicular with the diameter is the searched inverse point of $P'$.

Every line not containing the centre of inversion is transformed into a circle passing through the centre (figure 9.6) and reciprocally. To prove this statement let us take the polar equation of a straight line:

$$|FP| = \frac{d}{\cos\alpha}$$

Figure 9.6




Then the inverse of $P$ with centre $F$ has the equation:

$$|FP'| = \frac{k^2}{|FP|} = \frac{k^2\cos\alpha}{d}$$

which is the polar equation of a circle passing through $F$ and with diameter $k^2 / d$ . $\alpha$ is the angle between $FP$ and $FD$, the direction perpendicular to the line; therefore, the centre $O$ of the circle is located on $FD$.

Let us prove that an inversion transforms a circle not passing through its centre into another circle also not passing through its centre. Consider a circle of centre $F$ and radius $r$ which will be transformed under an inversion of centre $P$ and radius $k$. Then any point $R$ on the circle is mapped into another point $S$ according to:

$$|PS| = \frac{k^2}{|PR|} = \frac{k^2}{|PF|\cos\alpha \pm \sqrt{r^2 - PF^2\sin^2\alpha}}$$

where the polar equation of the circle is used. The multiplication by the conjugate of the denominator gives:

$$|PS| = \frac{k^2\left(|PF|\cos\alpha \mp \sqrt{r^2 - PF^2\sin^2\alpha}\right)}{PF^2 - r^2}$$

which is the polar equation of a circle with centre $G$ and radius $s$:

$$|PS| = |PG|\cos\alpha \mp \sqrt{s^2 - PG^2\sin^2\alpha}$$

where $$|PG| = \frac{k^2|PF|}{PF^2 - r^2} = \frac{k^2|PF|}{PT^2} \qquad s = \frac{k^2\, r}{PF^2 - r^2} = \frac{k^2\, r}{PT^2}$$

$PF^2 - r^2 = PT^2$ is the power of the point $P$ with respect to the circle of centre $F$. Then we see that the distance from the centre of inversion to the centre of the circle and the radius of this circle changes in the ratio of the square of the radius of inversion divided by the power of the centre of inversion with respect to the given circle.

What is the power of the centre of inversion $P$ with respect to the new circle?:

$$PU^2 = PG^2 - s^2 = \frac{k^4\left(PF^2 - r^2\right)}{PT^4} = \frac{k^4}{PT^2}$$

That is, the product of the powers of the centre of inversion with respect to any circle and its transformed is constant and equal to the fourth power of the radius of inversion:

$$PT^2\, PU^2 = k^4$$

In the equation of the transformed circle the symbol $\mp$ appears instead of $\pm$ . It means that the sense of the arc from $R$ to $R'$ is opposite to the sense of the arc from $S$ to $S'$.

### The nine-point circle

The Euler's theorem states that the midpoints of the sides of any triangle, the feet of the altitudes and the midpoints from each vertex to the orthocentre lie on a circle called the *nine-point circle*.

Let any triangle $PQR$ whose orthocentre is $H$ be. Now we evaluate the expression:

$$(P - H + Q - R)^2 - (P - H - Q + R)^2 = 4(P - H)\cdot(Q - R) = 0$$

which vanishes because the segment $PH$ lying on the altitude passing through $P$ is perpendicular to the base $QR$. Analogously:

$$(P - H + Q - R)^2 - (P + H - Q - R)^2 = 4(Q - H)\cdot(P - R) = 0$$

and

$$(P + H - Q - R)^2 - (P - H - Q + R)^2 = 4(R - H)\cdot(Q - P) = 0$$

Hence:
$$(P + Q - R - H)^2 = (P - Q + R - H)^2 = (P - Q - R + H)^2$$

Since the sign may be opposite inside the square we have:

$$(P + Q - R - H)^2 = (P - Q + R - H)^2 = (P - Q - R + H)^2 =$$

$$= (-P - Q + R + H)^2 = (-P + Q - R + H)^2 = (-P + Q + R - H)^2$$

Now we introduce the point $N = (P + Q + R + H)/4$ to obtain:

$$\left(N - \frac{R+H}{2}\right)^2 = \left(N - \frac{Q+H}{2}\right)^2 = \left(N - \frac{Q+R}{2}\right)^2 = \left(N - \frac{P+Q}{2}\right)^2 = \left(N - \frac{P+R}{2}\right)^2 = \left(N - \frac{P+H}{2}\right)^2$$

That is, $N$ is the centre of a circle (figure 9.7) passing through the three midpoints of the sides $(P + Q)/2$, $(Q + R)/2$ and $(R + P)/2$ and the three midpoints of the vertices and the orthocentre $(P + H)/2$, $(Q + H)/2$ and $(R + H)/2$.

Since we can write:

$$N = \frac{1}{2}\left(\frac{P+H}{2}\right) + \frac{1}{2}\left(\frac{Q+R}{2}\right)$$

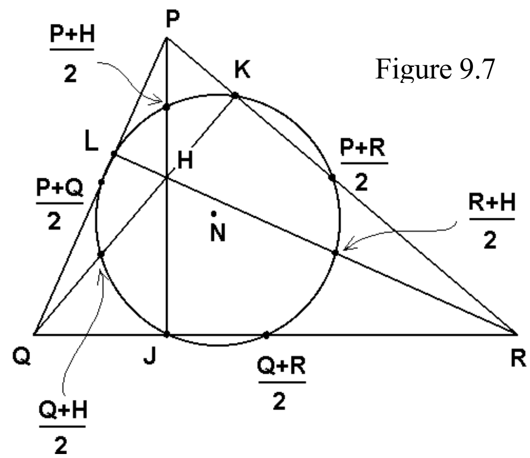



Figure 9.7

it follows that $(P+H)/2$ and $(Q+R)/2$ are opposite extremes of the same diameter. Then the vectors $J-(P+H)/2$ and $J-(Q+R)/2$, which are orthogonal:

$$\left(J - \frac{P+H}{2}\right) \cdot \left(J - \frac{Q+R}{2}\right) = 0$$

are the sides of a right angle which intercepts a half circumference of the nine point circle. Therefore, this angle is inscribed and its vertex $J$ also lies on the nine-point circle. Also, we may go through the algebraic way to arrive at the same conclusion. Developing the former inner product:

$$J^2 - J \cdot \frac{Q+R}{2} - \frac{P+H}{2} \cdot J + \frac{P+H}{2} \cdot \frac{Q+R}{2} = 0$$

Introducing the centre $N$ of the nine-point circle, we have:

$$J^2 - 2\,J \cdot N + 2\,N \cdot \frac{Q+R}{2} - \left(\frac{Q+R}{2}\right)^2 = 0$$

and after adding and subtracting $N^2$ we arrive at:

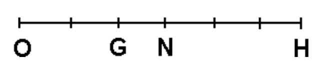
$$(J-N)^2 - \left(\frac{Q+R}{2} - N\right)^2 = 0$$

which shows that the length of $JN$ is the radius of the nine-point circle.

From the formulas for the centroid and orthocentre one obtains the centre $N$ of the nine-point circle:

$$N = \frac{3G+H}{4} = \left(P\,QR\,P + Q\,RP\,Q + R\,PQ\,R\right)\left(4PQ \wedge QR\right)^{-1}$$

Figure 9.8



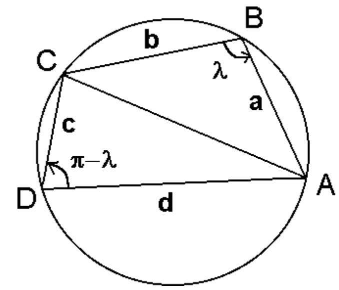

that is, the point $N$ also lies on the Euler's line. The relative distances between the circumcentre $O$, the centroid $G$, the nine-point circle centre $N$ and the orthocentre $H$ are shown in the figure 9.8.

### Cyclic and circumscribed quadrilaterals

First, let us see a statement valid for every quadrilateral: the area of a quadrilateral *ABCD* is the outer product of the diagonals *AC* and *BD*. The prove is written in one line. Since we can divide the quadrilateral in two triangles we have:

$$\begin{aligned}\text{Area} &= \frac{1}{2} AB \wedge BC + \frac{1}{2} AC \wedge CD \\ &= \frac{1}{2}(AC \wedge BC + AC \wedge CD) \\ &= \frac{1}{2} AC \wedge BD \\ &= \frac{1}{2}\,|AC|\,|BD|\sin(AC, BD)\, e_{12}\end{aligned}$$

Figure 9.9




The quadrilaterals inscribed in a circle are called *cyclic quadrilaterals*. They fulfil the Ptolemy's theorem: the product of the lengths of both diagonals is equal to the addition of the products of the lengths of opposite sides (figure 9.9):

$$|AC||BD| = |AB||CD| + |BC||DA|$$

A beautiful demonstration makes use of the cross ratio and it is proposed in the first exercise of the next chapter.

For the cyclic quadrilaterals, the sum of opposite angles is equal to $\pi$, because they are inscribed in the circle and intercept opposite arcs. Another interesting property is the Brahmagupta formula for the area. If the lengths of the sides are denoted by $a$, $b$, $c$ and $d$ and the semiperimeter by $s$ then:

$$|\text{Area}| = \sqrt{(s-a)\,(s-b)\,(s-c)\,(s-d)} \qquad s = \frac{a+b+c+d}{2}$$

With this notation, the proof is written more briefly. The law of cosines applied to each triangle of quadrilateral gives:

$$\begin{cases} AC^2 = a^2 + b^2 - 2\,a\,b\cos\lambda \\ AC^2 = c^2 + d^2 - 2\,c\,d\cos(\pi - \lambda) \end{cases}$$

Equating both equations taking into account that the cosines of supplementary angles have opposite sign, we have:

$$a^2 + b^2 - 2\,a\,b\cos\lambda = c^2 + d^2 + 2\,c\,d\cos\lambda$$

$$a^2 + b^2 - c^2 - d^2 = 2\,(\,a\,b + c\,d\,)\cos\lambda$$

$$\frac{a^2+b^2-c^2-d^2}{2\left(a\,b+c\,d\right)}=\cos\lambda$$

The area of the quadrilateral is the sum of the areas of the triangles *ABC* and *CDA*:

$$|\,\text{Area}\,|=\frac{a\,b}{2}\sin\lambda+\frac{c\,d}{2}\sin(\pi-\lambda)$$

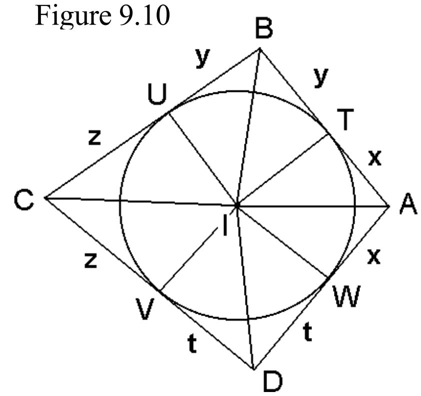
Both sines are equal; then we have:

$$\text{Area}^2=\frac{\left(a\,b+c\,d\right)^2}{4}\sin^2\lambda=\frac{\left(a\,b+c\,d\right)^2}{4}\left(1-\frac{\left(a^2+b^2-c^2-d^2\right)^2}{4\left(a\,b+c\,d\right)^2}\right)$$

$$=\frac{4\left(a\,b+c\,d\right)^2-\left(a^2+b^2-c^2-d^2\right)^2}{16}=(s-a)(s-b)(s-c)(s-d)$$

Another type of quadrilaterals are those where a circle can be inscribed, also called *circumscribed quadrilaterals*. For these quadrilaterals the bisectors of all the angles meet in the centre *I* of the inscribed circle, because it is equidistant from all the sides. When tracing the radii going to the tangency points, the quadrilateral is divided into pairs of opposite triangles (figure 9.10), and hence one deduces that the sums of the lengths of opposite sides are equal:

Figure 9.10




$$|\,AB\,|+|\,CD\,|=x+y+z+t=y+z+t+x=|\,BC\,|+|\,DA\,|$$

If a quadrilateral has inscribed and circumscribed circle, then we may substitute this condition ($a+c=b+d$) in the Brahmagupta formula to find:

$$|\,\text{Area}\,|=\sqrt{a\,b\,c\,d}=\sqrt{|\,AB\,|\,|\,BC\,|\,|\,CD\,|\,|\,DA\,|}$$

Since the sum of the angles of any quadrilateral is $2\pi$ and either of the four small quadrilaterals of a circumscribed quadrilateral has two right angles (figure 9.10), it follows that the central angles are supplementary of the vertices:

$$CBA\equiv UBT=\pi-TIU \qquad\qquad ADC\equiv WDV=\pi-VIW$$

If the quadrilateral is also cyclic then the angles *CBA* and *ADC* are inscribed and intercept opposite arcs of the outer circle, so that the segments joining opposite tangency points are orthogonal:

$$CBA + ADC = \pi \qquad \Rightarrow \qquad TIU + VIW = \pi \qquad \Rightarrow \qquad TV \perp UW$$

**Angle between circles**

Let us calculate the angle between a circle centred at $O$ with radius $r$ and another one centred at $O'$ with radius $r'$. If $|O' - O| < r + r'$ then they intersect. The points of intersection $P$ must fulfil the equations for both circles:

$$\begin{cases} (P-O)^2 = r^2 \\ (P-O')^2 = r'^2 \end{cases} \qquad \Rightarrow \qquad \begin{cases} P^2 - 2\,P \cdot O + O^2 = r^2 \\ P^2 - 2\,P \cdot O' + O'^2 = r'^2 \end{cases}$$

Adding both equations we obtain:

$$2\,P^2 - 2\,P \cdot (O + O') + O^2 + O'^2 = r^2 + r'^2$$

It is trivial that the angle of intersection is the supplementary of the angle between both radius, its cosines being equal:

$$\cos\alpha = \frac{OP \cdot O'P}{|OP|\,|OP'|} = \frac{P^2 - P \cdot (O + O') + O \cdot O'}{r\,r'}$$

The substitution of the first equality into the second gives:

$$\cos\alpha = \frac{r^2 + r'^2 - (O - O')^2}{2\,r\,r'}$$

The cosine of a right angle is zero; therefore two circles are orthogonal if and only if:

$$r^2 + r'^2 = (O - O')^2$$

which is the Pythagorean theorem.

**Radical axis of two circles**

The *radical axis* of two circles is the locus of the points which have the same power with respect to both circles:

$$OP^2 - r^2 = O'P^2 - r'^2$$

Let $X$ be the intersection of the radical axis with the line joining both centres of the circles. Then:

$$OX^2 - r^2 = O'X^2 - r'^2 \qquad \Rightarrow \qquad OX^2 - O'X^2 = r^2 - r'^2$$

$$(OX + O'X) \cdot (OX - O'X) = r^2 - r'^2$$

$$2\left( X - \frac{O+O'}{2} \right) \cdot OO' = r^2 - r'^2$$

Since $OX$ and $O'X$ are proportional to $OO'$, the inner and geometric products are equivalent and we may isolate $X$:

$$X = \frac{O + O' + \left(r^2 - r'^2\right) OO'^{-1}}{2}$$

Now let us search which kind of geometric figure is the radical axis. Arranging as before the terms, we have:

$$2\left( P - \frac{O+O'}{2} \right) \cdot OO' = r^2 - r'^2$$

Introducing the point $X$ we arrive at:

$$2\,(P - X) \cdot OO' = 0$$

That is, the radical axis is a line passing through $X$ and perpendicular to the line joining both centres. If the circles intersect, then the radical axis is the straight line passing through the intersection points.

The *radical centre* of three circles is the intersection of their radical axis. Let $O$, $O'$ and $O''$ be the centre of three circles with radii $r$, $r'$ and $r''$. Let us denote with $X$ (as above) the intersections of the radical axis of the first and second circles with the line $OO'$, and with $Y$ the intersection of the radical axis of the second and third circles with the line joining its centres $O'O''$. The radical centre $P$ must fulfil the former equation for both radical axis:

$$\begin{cases} (P - X) \cdot OO' = 0 \\ (P - Y) \cdot O'O'' = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} P \cdot OO' = X \cdot OO' = \dfrac{O'^2 - O^2 + r^2 - r'^2}{2} \\ P \cdot O'O'' = Y \cdot O'O'' = \dfrac{O''^2 - O'^2 + r'^2 - r''^2}{2} \end{cases}$$

Now we have encountered the coefficients of linear combination of an imaginary decomposition. This process is explained in page 74 for the calculation of the orthocentre. However there is a difference: we wish not to resolve the vector into orthogonal components but from the known coefficients we wish to reconstruct the vector $P$. Then:

$$P = \frac{P \cdot O'O''}{OO' \wedge O'O''} OO' - \frac{P \cdot OO'}{OO' \wedge O'O''} O'O''$$

The substitution of the known values of the coefficients results in the following formula for the radical centre which is invariant under cyclic permutation of the circles:

$$P = \left(2\, OO' \wedge O'O''\right)^{-1} \left(\left(O''^2 - r''^2\right) OO' + \left(O^2 - r^2\right) O'O'' + \left(O'^2 - r'^2\right) O''O\right)$$

**Exercises**

9.1 Let $A$, $B$, $C$ be the vertices of any triangle. The point $M$ moves according to the equation:

$$MA^2 + MB^2 + MC^2 = k$$

Which geometric locus does the point $M$ describe as a function of the values of the real parameter $k$?

9.2 Prove that the geometric locus of the points $P$ such that the ratio of distances from $P$ to two distinct points $A$ and $B$ is a constant $k$ is a circle. Calculate the centre and the radius of this circle.

9.3 Prove the Simson's theorem: the feet of the perpendiculars from a point $D$ upon the sides of a triangle $ABC$ are aligned if and only if $D$ lies on the circumscribed circle.

9.4 The Bretschneider's theorem: let $a$, $b$, $c$ and $d$ be the successive sides of a quadrilateral, $m$ and $n$ its diagonals and $\alpha$ and $\gamma$ two opposite angles. Show that the following *law of cosines for a quadrilateral* is fulfilled:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - 2|\,a\,||\,b\,||\,c\,||\,d\,|\cos(\alpha + \gamma)$$

9.5 Draw three circles passing through each vertex of a triangle and the midpoints of the concurrent sides. Then join the centre of each circle and the midpoint of the opposite side. Show that the three segments obtained in this way intersect in a unique point.

9.6 Prove that the inversion is an opposite conformal transformation, that is, it preserves the angles between curves, but changes their sign.

## 10. CROSS RATIOS AND RELATED TRANSFORMATIONS

### Complex cross ratio

The *complex cross ratio* of any four points *A*, *B*, *C* and *D* on the plane is defined as the quotient of the following two single ratios:

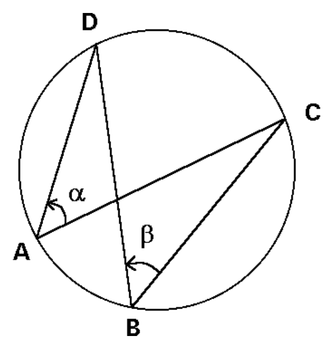
$$(\,A\,B\,C\,D\,) = (\,A\,C\,D\,)\,(\,B\,C\,D\,)^{-1} = AC\,AD^{-1}\,BD\,BC^{-1}$$

If the four points are aligned, then the cross ratio is a real number, otherwise it is a complex number. Denoting the angles *CAD* and *CBD* as $\alpha$ and $\beta$ respectively, the cross ratio is written as:

$$(\,A\,B\,C\,D\,) = \frac{AC\;AD\;BD\;BC}{|\,AD\,|^2\,|\,BC\,|^2}$$

$$= \frac{|\,AC\,|\,|\,BD\,|}{|\,AD\,|\,|\,BC\,|}\exp[(\alpha-\beta)e_{12}]$$

Figure 10.1



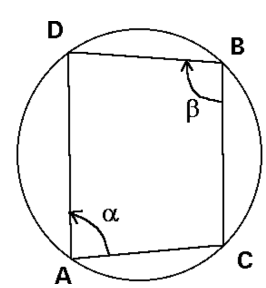

If the four points are located in this order on a circle (figure 10.1), the cross ratio is a positive real number, because the inscribed angles $\alpha$ and $\beta$ intercept the same arc *CD* and hence are equal, so:

$$(\,A\,B\,C\,D\,) = \frac{|\,AC\,|\,|\,BD\,|}{|\,AD\,|\,|\,BC\,|}$$

But if one of the points *C* or *D* is located between *A* and *B* (figure 10.2), then $\alpha$ and $\beta$ have distinct arcs *CD* with opposite orientation. Since $\alpha$ and $\beta$ are the half of these arcs, it follows that $\alpha - \beta = \pi$ and the cross ratio is a negative real number:

$$(\,A\,B\,C\,D\,) = -\frac{|\,AC\,|\,|\,BD\,|}{|\,AD\,|\,|\,BC\,|}$$

Figure 10.2




Analogously, if the four points are aligned and *C* or *D* is located between *A* and *B*, the cross ratio is negative[1].

Other definitions of the cross ratio differ from this one given here only in the order in which the vectors are taken. This question is equivalent to study how the value of the cross ratio changes under permutations of the points. For four points, there are 24 possible permutations, but only six different values for the cross ratio. The calculations are somewhat laborious and I shall only show one case.

[1] A straight line is a circle with infinite radius.

The reader may prove any other case as an exercise. I also indicate whether the permutations are even or odd, that is, whether they are obtained by an even or an odd number of exchanges of any two points from the initial definition $( A\,B\,C\,D )$:

(1) $( A\,B\,C\,D ) = ( B\,A\,D\,C ) = ( C\,D\,A\,B ) = ( D\,C\,B\,A ) = r$ even

(2) $( B\,A\,C\,D ) = ( A\,B\,D\,C ) = ( D\,C\,A\,B ) = ( C\,D\,B\,A ) = \dfrac{1}{r}$ odd

(3) $( A\,C\,B\,D ) = ( B\,D\,A\,C ) = ( C\,A\,D\,B ) = ( D\,B\,C\,A ) = 1 - r$ odd

(4) $( C\,A\,B\,D ) = ( D\,B\,A\,C ) = ( A\,C\,D\,B ) = ( B\,D\,C\,A ) = \dfrac{1}{1-r}$ even

(5) $( B\,C\,A\,D ) = ( A\,D\,B\,C ) = ( D\,A\,C\,B ) = ( C\,B\,D\,A ) = \dfrac{r-1}{r}$ even

(6) $( C\,B\,A\,D ) = ( D\,A\,B\,C ) = ( A\,D\,C\,B ) = ( B\,C\,D\,A ) = \dfrac{r}{r-1}$ odd

Let us prove, for example the equalities (3):

(3) $( A\,C\,B\,D ) = AB\,AD^{-1}\,CD\,CB^{-1} = ( AC + CB )\,AD^{-1}\,( CB + BD )\,CB^{-1} =$

$$= AC\,AD^{-1}\,CB\,CB^{-1} + AC\,AD^{-1}\,BD\,CB^{-1} + CB\,AD^{-1}\,CB\,CB^{-1} + CB\,AD^{-1}\,BD\,CB^{-1}$$

In the last term, under the reflection in the direction $CB$, the complex number $AD^{-1}\,BD$ becomes conjugate, that is, these vectors are exchanged:

$$= AC\,AD^{-1} + AC\,AD^{-1}\,BD\,CB^{-1} + CB\,AD^{-1} + BD\,AD^{-1}$$

$$= ( AC + CB + BD )\,AD^{-1} - AC\,AD^{-1}\,BD\,BC^{-1} = AD\,AD^{-1} - r = 1 - r$$

Let us see that this value is the same for all the cross ratios of the case (3):

$$( B\,D\,A\,C ) = BA\,BC^{-1}\,DC\,DA^{-1} = AB\,CB^{-1}\,CD\,AD^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= ( A\,C\,B\,D )$$

where in the third step the permutative property has been applied. Also by using this property one obtains:

$$( C\,A\,D\,B ) = CD\,CB^{-1}\,AB\,AD^{-1} = CD\,AD^{-1}\,AB\,CB^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= ( A\,C\,B\,D )$$

$$( D\,B\,C\,A ) = DC\,DA^{-1}\,BA\,BC^{-1} = BA\,DA^{-1}\,DC\,BC^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= ( A\,C\,B\,D )$$

**Harmonic characteristic and ranges**

From the cross ratio, one can find a quantity independent of the symbols of the points and only dependent on their locations. I call this quantity the *harmonic characteristic* -because of the obvious reasons that follow now- denoting it as [$A\ B\ C\ D$]. The simplest way to calculate it (but not the unique) is through the alternated addition of all the permutations of the cross ratio, condensed in the alternated sum of the six different values:

$$(A\,B\,C\,D) - (B\,A\,C\,D) - (A\,C\,B\,D) + (C\,A\,B\,D) + (B\,C\,A\,D) - (C\,B\,A\,D) =$$

$$= r - \frac{1}{r} - (1-r) + \frac{1}{1-r} + \frac{r-1}{r} - \frac{r}{r-1}$$

$$= \frac{(2\,r-1)(r+1)(r-2)}{r\,(1-r)}$$

When two points are permuted, this sum still changes the sign. The square of this sum is invariant under every permutation of the points, and is the suitable definition of the harmonic characteristic:

$$[A\,B\,C\,D] = \left[(A\,B\,C\,D) - (B\,A\,C\,D) - (A\,C\,B\,D) + (C\,A\,B\,D) + (B\,C\,A\,D) - (A\,B\,C\,D)\right]^2$$

$$[A\,B\,C\,D] = \frac{(2\,r-1)^2\,(r+1)^2\,(r-2)^2}{r^2\,(1-r)^2}$$

The first notable property of the harmonic characteristic is the fact that it vanishes for a harmonic range of points, that is, when the cross ratio ($A\ B\ C\ D$) takes the values 1/2, –1 or –2 dependent on the denominations of the points. The second notable property is the fact that the harmonic characteristic of a degenerate range of points (two or more coincident points) is infinite.

Perhaps, the reader believes that other harmonic characteristics may be obtained in many other ways, that is, by using other combinations of the permutations of the cross ratio. However, other attempts lead whether to the same function of $r$ (or a linear dependence) or to constant values. For example, we can take the sum of all the squares, which is also invariant under any permutation of the points to find:

$$(A\,B\,C\,D)^2 + (B\,A\,C\,D)^2 + (A\,C\,B\,D)^2 + (C\,A\,B\,D)^2 + (B\,C\,A\,D)^2 + (C\,B\,A\,D)^2$$

$$= r^2 + \frac{1}{r^2} + (1-r)^2 + \frac{1}{(1-r)^2} + \frac{(r-1)^2}{r^2} + \frac{r^2}{(r-1)^2} = \frac{[A\,B\,C\,D] + 21}{2}$$

On the other hand, M. Berger (vol. I, p. 127) has used a function that is also linearly dependent on the harmonic characteristic:

$$\frac{4\left(r^2-r+1\right)^3}{27\,r^2\left(1-r\right)^2}=1+\frac{[A\,B\,C\,D]}{27}$$

Other useless possibilities are the sum of all the values of the permutations of the cross ratio, which is identical to zero, or their product giving always a unity result.

Four points $A$, $B$, $C$ and $D$ so ordered on a line are said to be a *harmonic range* if the segments $AB$, $AC$ and $AD$ are in harmonic progression, that is, the inverse vectors $AB^{-1}$, $AC^{-1}$ and $AD^{-1}$ are in arithmetic progression:

$$AB^{-1}-AC^{-1}=AC^{-1}-AD^{-1} \qquad \Leftrightarrow \qquad AC\,AD^{-1}\,BD\,BC^{-1}=2$$

The cross ratio of a harmonic range is 2, but if this order is altered then it can be –1 or 1/2. Note that this definition also includes harmonic ranges of points on a circle. On the other hand, all the points of a harmonic range always lie on a line or a circle.

In order to construct the fourth point $D$ forming a harmonic range with any three given points $A$, $B$ and $C$ (ordered in this way on a line) one must follow these steps (figure 10.3):

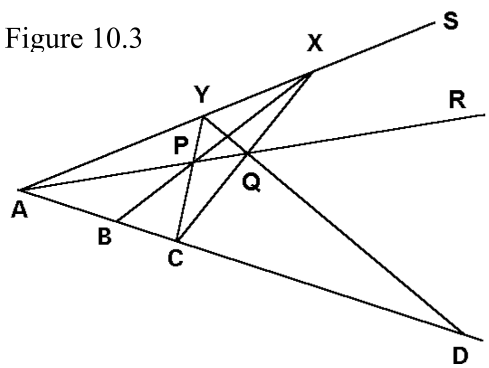
1) Draw the line passing through $A$, $B$ and $C$. Trace two any distinct lines $R$ and $S$ also different from $ABC$ and passing through $A$.

2) Draw any line passing through $B$. This line will cut the line $R$ in the point $P$ and the line $S$ in the point $X$.

3) Draw the line passing through $C$ and $X$. The intersection of this line with $R$ will be denoted by $Q$. Draw also the line passing through $C$ and $P$. This line intersects the line $S$ in the point $Y$.

4) Now trace the line passing through $Y$ and $Q$, which will cut the line $ABC$ in the searched point $D$, which completes the harmonic range.

Figure 10.3




For a clearer displaying I have drawn some segments instead of lines, although the construction is completely general only for lines.

If $A$, $B$ and $C$ are not aligned, the point $D$ which completes the harmonic range lies on a circle passing through the three given points (figure 10.4). In order to determine $D$ one must make an inversion (which preserves the cross ratio if this is real, as shown below) with centre on the circle $ABC$, which transforms this circle into a line. In this line we determine the point $D'$ forming a harmonic range with $A'$, $B'$, $C'$ and then, by means of the same inversion, the point $D$ is obtained. In the drawing method the radius and the exact position of the centre of inversion do not play any role whenever it lie on the circle $ABC$. Then we shall draw any line outside the circle, project $A$, $B$ and $C$ into this line, make the former construction and obtain $D$ by projecting $D'$ upon the circle.

The cross ratio becomes conjugated under circular inversion. In order to prove this statement, let us see firstly how the single ratio is transformed. If $A'$, $B'$, $C'$ and $D'$ are the transformed points of $A$, $B$, $C$ and $D$ under an inversion with centre $O$ and any radius, we have:

$$A'C'\,A'D'^{-1} = (OC' - OA')\,(OD' - OA')^{-1} = (OC^{-1} - OA^{-1})\,(OD^{-1} - OA^{-1})^{-1}$$

$$= OC^{-1}\,(\,OA - OC\,)\,OA^{-1}\,[\,OD^{-1}\,(\,OA - OD\,)\,OA^{-1}\,]^{-1}$$

$$= OC^{-1}\,CA\,OA^{-1}\,[\,OD^{-1}\,DA\,OA^{-1}\,]^{-1} = OC^{-1}\,CA\,OA^{-1}\,OA\,DA^{-1}\,OD$$

$$= OC^{-1}\,AC\,AD^{-1}\,OD$$

Analogously we have:

$$B'D'\,B'C'^{-1} = OD^{-1}\,BD\,BC^{-1}\,OC$$

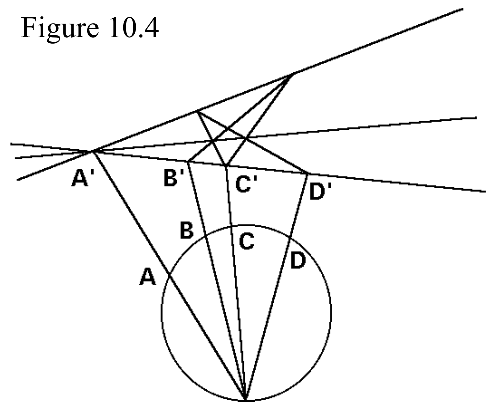
The product of both expressions is the cross ratio:

$$(\,A'\,B'\,C'\,D'\,) = A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}$$

$$= OC^{-1}\,AC\,AD^{-1}\,BD\,BC^{-1}\,OC$$

$$= OC^{-1}\,(\,A\,B\,C\,D\,)\,OC = (\,A\,B\,C\,D\,)^*$$

Figure 10.4




So, the cross ratio of four points on a line or a circle, which is real, is preserved, otherwise it becomes conjugated.

Finally let us see that the single ratio is the limit of the cross ratio for one point at infinity:

$$(\,\infty\,B\,C\,D\,) = \lim_{A\to\infty} AC\,AD^{-1}\,BD\,BC^{-1} = BD\,BC^{-1} = (\,B\,D\,C\,)$$

Analogously $(\,A\,\infty\,C\,D\,) = (\,A\,C\,D\,)$, etc.

### The homography (Möbius transformation)

The *homography* is defined as the geometric transformation that preserves the complex cross ratio of four points on the plane. In order to specify a homography one must gives only the images $A'$, $B'$ and $C'$ of three points $A$, $B$, and $C$ because the fourth image $D'$ is determined by the conservation of their cross ratio:

$$(\,A'\,B'\,C'\,D'\,) = (\,A\,B\,C\,D\,)$$

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = AC\,AD^{-1}\,BD\,BC^{-1}$$

Arranging vectors we write:

$$A'D'^{-1}\,B'D' = A'C'^{-1}\,AC\,AD^{-1}\,BD\,BC^{-1}\,B'C' = A'C'^{-1}\,AC\;\,BC^{-1}\,B'C'\,AD^{-1}\,BD$$

After conjugation we find:

$$B'D'\, A'D'^{-1} = BD\, AD^{-1}\, B'C'\, BC^{-1}\, AC\, A'C'^{-1}$$

$$B'D'\, A'D'^{-1} = BD\, AD^{-1}\, z \qquad \text{where} \quad z = BC^{-1}\, B'C'\, A'C'^{-1}\, AC$$

$$D'B'\, D'A'^{-1} = DB\, DA^{-1}\, z \quad \Rightarrow \quad (\, D'\, B'\, A'\,) = (\, D\, B\, A\,)\, z$$

Hence the homography multiplies the single ratio by a complex number $z$. The modulus of the single ratio increases proportionally to the modulus of $z$ and the angle $BDA$ is increased in the argument of $z$:

$$z = |\, z\,| \;\exp[\alpha\, e_{12}\,]$$

$$\frac{|\, D'B'\,|}{|\, D'A'\,|} = |\, z\,| \frac{|\, DB\,|}{|\, DA\,|}$$

$$\text{angle } B'D'A' = \text{angle } BDA + \alpha$$

Then a homography can also be determined by specifying the images $A'$ and $B'$ of two points $A$ and $B$, and a complex number $z$. The image $D'$ of $D$ is calculated by isolation in the former equation:

$$(\, D'A' + A'B'\,)\, D'A'^{-1} = DB\, DA^{-1}\, z$$

$$A'B'\, D'A'^{-1} = DB\, DA^{-1}\, z - 1$$

$$D'A' = (\, z\, DB\, DA^{-1} - 1\,)^{-1}\, A'B'$$

$$D' = A' + (\, 1 - z\, DB\, DA^{-1}\,)^{-1}\, A'B'$$

The homography preserves the complex cross ratio of any four points. Analogously to the former equality, we find for any three points $P$, $Q$, $R$ :

$$P' = A' + (\, 1 - z\, PB\, PA^{-1}\,)^{-1}\, A'B'$$

$$Q' = A' + (\, 1 - z\, QB\, QA^{-1}\,)^{-1}\, A'B'$$

$$R' = A' + (\, 1 - z\, RB\, RA^{-1}\,)^{-1}\, A'B'$$

$$D'Q' = [\, (\, 1 - z\, QB\, QA^{-1}\,)^{-1} - (\, 1 - z\, DB\, DA^{-1}\,)^{-1}\,]\, A'B' =$$

$$= z\,(\, -DB\, DA^{-1} + QB\, QA^{-1}\,)\,(\, 1 - z\, DB\, DA^{-1}\,)^{-1}\,(\, 1 - z\, QB\, QA^{-1}\,)^{-1}\, A'B'$$

$$D'R' = z\,(\, -DB\, DA^{-1} + RB\, RA^{-1}\,)\,(\, 1 - z\, DB\, DA^{-1}\,)^{-1}\,(\, 1 - z\, RB\, RA^{-1}\,)^{-1}\, A'B'$$

$$P'Q' = z\,(\, -PB\, PA^{-1} + QB\, QA^{-1}\,)\,(\, 1 - z\, PB\, PA^{-1}\,)^{-1}\,(\, 1 - z\, QB\, QA^{-1}\,)^{-1}\, A'B'$$

$$P'R' = z\,(\,-PB\;PA^{-1} + RB\;RA^{-1}\,)\,(\,1 - z\;PB\;PA^{-1}\,)^{-1}\,(\,1 - z\;RB\;RA^{-1}\,)^{-1}\,A'B'$$

In the cross ratio, many factors are simplified:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = (\,-DB\;DA^{-1} + QB\;QA^{-1}\,)\,(\,-DB\;DA^{-1} + RB\;RA^{-1}\,)^{-1}$$

$$(\,-PB\;PA^{-1} + RB\;RA^{-1}\,)\,(\,-PB\;PA^{-1} + QB\;QA^{-1}\,)^{-1}$$

Writing the first factor as product of factors we obtain:

$$-DB\;DA^{-1} + QB\;QA^{-1} = -\,(\,DA + AB\,)\,DA^{-1} + (\,QA + AB\,)\,QA^{-1} =$$

$$= -AB\;DA^{-1} + AB\;QA^{-1} = AB\;DA^{-2}\,(\,-DA\;QA^2 + DA^2\;QA\,)\,QA^{-2} =$$

$$= AB\;DA^{-2}\;DA\,(\,-QA + DA\,)\,QA\;QA^{-2}$$

$$= AB\;DA^{-1}\;QD\;QA^{-1}$$

In the same way each of the other three factors can be written as product of factors:

$$-DB\;DA^{-1} + RB\;RA^{-1} = AB\;DA^{-1}\;RD\;RA^{-1}$$

$$-PB\;PA^{-1} + RB\;RA^{-1} = AB\;PA^{-1}\;RP\;RA^{-1}$$

$$-PB\;PA^{-1} + QB\;QA^{-1} = AB\;PA^{-1}\;QP\;QA^{-1}$$

Then the cross ratio will be:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = (\,AB\;DA^{-1}\;QD\;QA^{-1}\,)\,(\,RA\;RA^{-1}\;DA\;AB^{-1}\,)$$

$$(\,AB\;PA^{-1}\;RP\;RA^{-1}\,)\,(\,QA\;QP^{-1}\;PA\;AB^{-1}\,)$$

where the brackets are not needed but only show the factors they come from. Applying the permutative property and simplifying, we arrive at:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = DQ\;DR^{-1}\;PR\;PQ^{-1}$$

This proves that the cross ratio of every set of four point is preserved under a homography. Four points lying on a line or a circle have a real cross ratio. Then the homography is a circular transformation since it transforms circles into circles in a general sense (taking the lines as circles with infinite radius).

The product of two circular inversions is always a homography because the cross ratio becomes conjugate under inversion and thus it is preserved under an even number of inversions. Let us calculate the homography resulting from a composition of two inversions. Let $A$ be any point, $A'$ be the transformed point under a first inversion with centre $O$ and radius $r$, and $A''$ be the transformed point of $A'$ under a second inversion with centre $P$ and radius $s$, which is consequently the transformed point of $A$ under the homography:

$$OA' = r^2\, OA^{-1} \quad PA'' = s^2\, PA'^{-1}$$

From where it follows that:

$$s^2\, PA''^{-1} = PA' = OA' - OP = r^2\, OA^{-1} - OP$$

$$PA''^{-1} = s^{-2}\, (\, r^2\, OA^{-1} - OP\,)$$

$$PA'' = s^2\, (r^2\, OA^{-1} - OP\,)^{-1} = s^2\, [(\, r^2 - OP\, OA\,)\, OA^{-1}]^{-1} = s^2\, OA\, (r^2 - OP\, OA\,)^{-1}$$

This equation allows to determine the image $A''$ of any point $A$ having the centre $O$ and $P$ and the radius $r$ and $s$ as data. For instance, let us calculate the homography corresponding to:

$$O = (\,0, 0\,) \qquad r = 2 \qquad P = (\,3, 0\,) \qquad s = 3$$

$$OP = 3\, e_1$$

In order to specify the homography we determine the images of three points:

$$A = (\,0, 2\,) \qquad B = (\,1, 1\,) \qquad C = (\,1, 0\,)$$

$$OA = 2\, e_2 \qquad OB = e_1 + e_2 \qquad OC = e_1$$

$$OP\, OA = 6\, e_{12} \qquad OP\, OB = 3 + 3\, e_{12} \qquad OP\, OC = 3$$

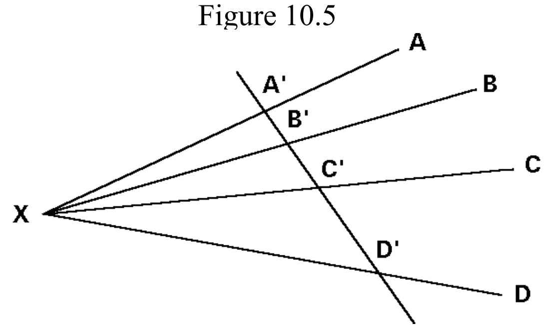
$$PA'' = \frac{-27\, e_1 + 18\, e_2}{13} \qquad PB'' = \frac{-9\, e_1 + 18\, e_2}{5} \qquad PC'' = 9\, e_1$$

$$A'' = \left(\frac{12}{13}, \frac{18}{13}\right) \qquad B'' = \left(\frac{6}{5}, \frac{18}{5}\right) \qquad C'' = (12, 0)$$

**Projective cross ratio**

The *projective cross ratio* of four points $A$, $B$, $C$ and $D$ with respect to a point $X$ is defined as the cross ratio of the pencil of lines $XA$, $XB$, $XC$ and $XD$, which is the cross ratio of the intersection points of these lines with any crossing line (figure 10.5). As it will be shown, the projective cross ratio is independent of the crossing line The intersection points $A'$, $B'$, $C'$ and $D'$ are

Figure 10.5



the projections with centre $X$ of the points $A$, $B$, $C$ and $D$ upon the crossing line. Due to the alignment of these projections, the projective cross ratio is always a real number:

$$\{ X, A\,B\,C\,D \} = (\,A'\,B'\,C'\,D'\,) = A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}$$

The projected points $A'$, $B'$, $C'$ and $D'$ are dependent on the crossing line and this expression is not suitable for calculations. Note that the segment $A'C'$ is the base of the triangle $XA'C'$, $B'D'$ is the base of the triangle $XB'D'$, etc. Since the altitudes of all these triangles are the distance from $X$ to the crossing line, their bases are proportional to their areas, so we can write:

$$\{X, A\,B\,C\,D\} = \frac{XA' \wedge A'C'\; XB' \wedge B'D'}{XA' \wedge A'D'\; XB' \wedge B'C'}$$

The area of a triangle is the half of the outer product of any two sides. For example for the triangle $XA'C'$ we have:

$$XA' \wedge A'C' = XA' \wedge (\,A'X + XC'\,) = XA' \wedge XC'$$

Then, the cross ratio becomes:

$$\{X, A\,B\,C\,D\} = \frac{XA' \wedge XC'\; XB' \wedge XD'}{XA' \wedge XD'\; XB' \wedge XC'}$$

The simplification of the modulus gives:

$$\{X, A\,B\,C\,D\} = \frac{\sin A'XC' \sin B'XD'}{\sin A'XD' \sin B'XC'} = \frac{\sin AXC \sin BXD}{\sin AXD \sin BXC}$$

because the angles $A'XC'$ and $AXC$ are identical, $B'XD'$ and $BXD$ are also identical, etc. Now it is obvious that the projective cross ratio does not depend on the crossing line, but only on the angles between the lines of the pencil. If we introduce the modulus of $XA$, $XB$, etc., we find the outer products of these vectors:

$$\{X, A\,B\,C\,D\} = \frac{XA \wedge XC\; XB \wedge XD}{XA \wedge XD\; XB \wedge XC}$$

Now, the arbitrary crossing line of the pencil has disappeared in the formula, which has become suitable for calculations with Cartesian coordinates[2]. When the points $A$, $B$, $C$ and $D$ are aligned, the complex and projective cross ratios have the same value independently of the point $X$, because the segments $AC$, $BD$, $AD$ and $BC$ have the same direction, which can be taken as the crossing line:

---

[2] The reader will find an application of this formula for the projective cross ratio to the error calculus in the electronic article "Estimation of the error committed in determining the place from where a photograph was taken", Ramon González, Josep Homs, Jordi Solsona (1999) http://www.terra.es/personal/rgonzal1 (translation of the paper published in the *Butlletí de la Societat Catalana de Matemàtiques* **12** [1997] 51-71).

$$\{X, A\,B\,C\,D\} = (\,A\,B\,C\,D\,)$$

If $A$, $B$, $C$ and $D$ lie on a circle, the complex cross ratio is equal to the projective cross ratio taken from any point $X$ on that circle. To prove this statement, make an inversion with centre $X$. Since the circle $ABCDX$ passes through the centre of inversion, it is transformed into a line, where the points $A'$, $B'$, $C'$ and $D'$ (transformed of $A$, $B$, $C$ and $D$ by the inversion) lie with the same cross ratio. Now observe that $X$, $A$ and $A'$ are aligned, also $X$, $B$ and $B'$, etc. So the projective cross ratio is equal to the complex cross ratio.

From the arguments given above, it follows that the cross ratio of a pencil of lines is the quotient of the outer products of their direction (or normal) vectors independently of their modulus:

$$\{X, A\,B\,C\,D\} = \frac{v_{XA} \wedge v_{XC}\; v_{XB} \wedge v_{XD}}{v_{XA} \wedge v_{XD}\; v_{XB} \wedge v_{XC}} = \frac{n_{XA} \wedge n_{XC}\; n_{XB} \wedge n_{XD}}{n_{XA} \wedge n_{XD}\; n_{XB} \wedge n_{XC}}$$

Instead of $XA$, $XB$, $XC$ and $XD$, I shall denote by $A$, $B$, $C$ and $D$ the pencil of lines passing through $X$. Then the projective cross ratio $\{X, ABCD\}$ is written as $(ABCD)$. Let us suppose that these lines have the following dual coordinates expressed in the lines base $\{O, P, Q\}$:

$$A = [x_A\,, y_A] \qquad B = [x_B\,, y_B] \qquad C = [x_C\,, y_C] \qquad D = [x_D\,, y_D]$$

Then the direction vector of the line $A$ is:

$$v_A = (1 - x_A - y_A)v_O + x_A v_P + y_A v_Q$$

But the direction vectors of the base fulfil:

$$v_O + v_P + v_Q = 0$$

Hence:

$$v_A = (-1 + 2x_A + y_A)v_P + (-1 + x_A + 2y_A)v_Q$$

and analogously:

$$v_B = (-1 + 2x_B + y_B)v_P + (-1 + x_B + 2y_B)v_Q$$

$$v_C = (-1 + 2x_C + y_C)v_P + (-1 + x_C + 2y_C)v_Q$$

$$v_D = (-1 + 2x_D + y_D)v_P + (-1 + x_D + 2y_D)v_Q$$

The outer product of $v_A$ and $v_C$ is:

$$v_A \wedge v_C = \left(x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A)\right) v_P \wedge v_Q$$

which leads to the fact that the cross ratio only depends on the dual coordinates:

$$(A\,B\,C\,D) = \frac{v_A \wedge v_C \; v_B \wedge v_D}{v_A \wedge v_D \; v_B \wedge v_C} =$$

$$= \frac{\left(x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A)\right)\left(x_D - x_B - (y_D - y_B) + 3(x_B\, y_D - x_D\, y_B)\right)}{\left(x_D - x_A - (y_D - y_A) + 3(x_A\, y_D - x_D\, y_A)\right)\left(x_C - x_B - (y_C - y_B) + 3(x_B\, y_C - x_C\, y_B)\right)}$$

$$= \frac{(x_A - 1/3, y_A - 1/3) \wedge (x_C - 1/3, y_C - 1/3)\;(x_B - 1/3, y_B - 1/3) \wedge (x_D - 1/3, y_D - 1/3)}{(x_B - 1/3, y_B - 1/3) \wedge (x_C - 1/3, y_C - 1/3)\;(x_A - 1/3, y_A - 1/3) \wedge (x_D - 1/3, y_D - 1/3)}$$

$$= \frac{(x_A - x_C, y_A - y_C) \wedge (x_C - 1/3, y_C - 1/3)\;(x_B - x_D, y_B - y_D) \wedge (x_D - 1/3, y_D - 1/3)}{(x_B - x_C, y_B - y_C) \wedge (x_C - 1/3, y_C - 1/3)\;(x_A - x_D, y_A - y_D) \wedge (x_D - 1/3, y_D - 1/3)}$$

And taking into account the fact that the dual points corresponding to the lines of the pencil are aligned in the dual plane, it follows that:

$$(A\,B\,C\,D) = (x_A - x_C, y_A - y_C)(x_B - x_C, y_B - y_C)^{-1}(x_B - x_D, y_B - y_D)(x_A - x_D, y_A - y_D)^{-1}$$

$$= AC\; BC^{-1}\; BD\; AD^{-1} = AC\; AD^{-1}\; BD\; BC^{-1}$$

where the four dual vectors are proportional. That is: the projective cross ratio of a pencil of lines is the cross ratio of the corresponding dual points. Obviously, by means of the duality principle the dual proposition must also hold: the cross ratio of four aligned points is equal to the projective cross ratio of the dual pencil of the corresponding lines on the dual plane.

### The points at the infinity and homogeneous coordinates

The points at the infinity play the same role as the points with finite coordinates in the projective geometry. Under a projectivity they can be mapped mutually without special distinction. In order to avoid the algebraic inconsistencies of the infinity, the *homogeneous coordinates* are defined as proportional to the barycentric coordinates with a no fixed homogeneous constant. The following expressions are equivalent and represent the same point:

$$(x,\; y) = (1 - x - y,\; x,\; y) = k\,(1 - x - y,\; x,\; y)$$

Then for instance:

$$(2, 3) = (-4, 2, 3) = (-20, 10, 15)$$

A point has finite coordinates if and only if the sum of the homogeneous coordinates are different from zero. In this case we can normalise the coordinates dividing by this sum in order to obtain the barycentric coordinates, and hence the Cartesian coordinates by omitting the first coordinate. On the other hand, the sum of the homogeneous coordinates sometimes vanishes and then they cannot be normalised (the normalisation implies a division by zero yielding infinite values). In this case, we are regarding a point at the infinity, whose algebraic operations should be performed with the finite values of the homogeneous coordinates[3]. For example:

$$(2, -1, -1) = (2\,\infty, -\infty, -\infty) = (\infty, \infty)$$

This is the point at the infinity in the direction of the bisector of the first quadrant. On the other hand, points at the infinity are equivalent to directions and hence to vectors. For example:

$$v = 2\,O - P - Q = -OP - OQ = -2\,e_1 - 2\,e_2$$

Every vector proportional to $v$ also represents this direction and the point at the infinity given above.

### Perspectivity and projectivity

A *perspectivity* is a central projection of a range of points on a line upon another line. Since the cross ratio of a pencil of lines is independent of the crossing line, it follows from the construction (figure 10.6) that a perspectivity preserves the projective cross ratio:

$$\{O, A\,B\,C\,D\} = \{O, A'\,B'\,C'\,D'\}$$

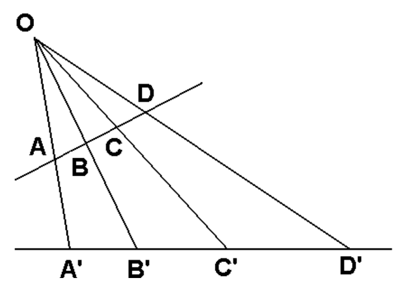



Figure 10.6

The composition of two perspectivities with different centres also maps ranges[4] of points to other ranges preserving their cross ratio. This is the most general transformation which preserves alignment and the projective cross ratio. A *projectivity* (also *projective transformation*) is defined as the geometric transformation which maps aligned points to aligned points preserving their cross ratio. A theorem of projective geometry states that any projectivity is expressible as the sequence of not more than three perspectivities. If the projectivity relates ranges on two distinct lines, two perspectivities suffice.

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[3] In fact, the calculations are equally feasible with $\infty$ if we understand it strictly as a number. Then the equality $2\infty = \infty$ should be wrong. We could simplify by $\infty$, e.g.: $(2\,\infty, -\infty, 1) = (2, -1, 0)$ is a finite point and $(2\,\infty, -\infty, -\infty) = (2, -1, -1)$ a point at the infinity.

[4] A *range* of points is defined as a set of aligned points.

A projectivity is determined by giving three aligned points and their corresponding images, which must be also aligned. On the other hand, a projectivity is determined by giving four points no three of which are collinear (a quadrilateral) and their non-aligned images (another quadrilateral). Since the projectivity preserves the alignment and the incidence, we must take a linear transformation of the coordinates. However, the affinity is already a linear transformation of coordinates but only of the independent coordinates. The more general linear transformation must account for all the three barycentric coordinates. Thus the projectivity is given by a 3×3 matrix:

$$\begin{pmatrix} 1-x'-y' \\ x' \\ y' \end{pmatrix} = k \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} \qquad \text{with } \det h_{ij} \neq 0$$

The coefficients $h_{ij}$ being given, the product of the matrices on the right hand side does not warrantee a set of coordinates with sum equal to 1. This means that the matrix $h_{ij}$ of a perspectivity is defined except by the proportionality constant $k$ (which depends on the coordinates), and the transformed point is obtained with homogeneous coordinates instead of barycentric ones. Nevertheless, this matrix maps collinear points to collinear points. This follows immediately from the matrices equalities:

$$\begin{pmatrix} k\,(1-x_A'-y_A') & l\,(1-x_B'-y_B') & m\,(1-x_C'-y_C') \\ k\,x_A' & l\,x_B' & m\,x_C' \\ k\,y_A' & l\,y_B' & m\,y_C' \end{pmatrix} = \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1-x_A-y_A & 1-x_B-y_B & 1-x_C-y_C \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix}$$

A set of points are collinear if the determinant of the barycentric coordinates vanishes. Since $\det h_{ij} \neq 0$, the implication of the alignment is bi-directional:

$$k\,l\,m \begin{vmatrix} 1-x_A'-y_A' & 1-x_B'-y_B' & 1-x_C'-y_C' \\ x_A' & x_B' & x_C' \\ y_A' & y_B' & y_C' \end{vmatrix} = \begin{vmatrix} 1-x_A-y_A & 1-x_B-y_B & 1-x_C-y_C \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{vmatrix} \det h_{ij}$$

As an example, let us transform the origin of coordinates under the following projectivity:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad (0,0) \to (1,2,1) = \left( \frac{1}{4}, \frac{2}{4}, \frac{1}{4} \right) = \left( \frac{1}{2}, \frac{1}{4} \right)$$

Analogously the transformed point of (1, 0) is:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \Rightarrow \quad (1, 0) \to (-1, 3, 2) = \left(\frac{3}{4}, \frac{1}{2}\right)$$

Also we may transform the point $(0, \infty)$ at the infinity, which has the homogeneous coordinates $(-1, 0, 1)$:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad (0, \infty) \to (1, -1, -1) = (1, 1)$$

This case exemplifies how points at the infinity can be mapped under a projectivity to points with finite coordinates and vice versa. As a comparison, the affinity always maps points at the infinity to points at the infinity (although it does not leave them invariant). That is, the affinity always maps parallel lines to parallel lines, while under a projectivity parallel lines can be mapped to concurrent lines and vice versa[5]. Parallel lines are concurrent in a point at the infinity, which does not play any special role in the projectivity.

Let us see that four non collinear points (a quadrilateral) and their non collinear images (another quadrilateral) determine a unique projectivity. Let us suppose that the images of the points (0, 0), (0, 1), (1, 0) and (1, 1) are $(a_1, a_2)$, $(b_1, b_2)$, $(c_1, c_2)$ and $(d_1, d_2)$ where no three of which are collinear. The barycentric coordinates of these points are:

$$(0, 0) = (1, 0, 0) \qquad (1, 0) = (0, 1, 0) \qquad (0, 1) = (0, 0, 1)$$

$$(a_1, a_2) = (a_0, a_1, a_2) \quad \text{with} \quad a_0 = 1 - a_1 - a_2 \qquad \text{etc.}$$

The substitution into the matrix equality produces a system of three equations with three unknowns $k$, $l$ and $m$:

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} k\,a_0 & l\,b_0 & m\,c_0 \\ k\,a_1 & l\,b_1 & m\,c_1 \\ k\,a_2 & l\,b_2 & m\,c_2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Since the points (0, 0), (1, 0) and (0, 1) are not collinear, the matrix is regular and the system has a unique solution, which proves that a projectivity is determined by the image of the base quadrilateral. If the vertices of the initial quadrilateral are not the base points, the matrix is calculated as the product of the inverse matrix of the projectivity which maps the base quadrilateral to the initial quadrilateral multiplied by the matrix of the projectivity which maps the point base to the final quadrilateral.

Let us calculate the projectivity which maps the quadrilateral with vertices (1, 1)-(2, 3)-(3, 1)-(1, 0) to the quadrilateral (4, 1)-(6, 0)-(4, 0)-(6, 1). The projectivity which

[5] Of course, the affinity is a special kind of projectivity.

maps the base quadrilateral (0, 0)-(1, 0)-(0, 1)-(1/3, 1/3) to the first quadrilateral is given by:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -k & -4\,l & -3\,m \\ k & 2\,l & 3\,m \\ k & 3\,l & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

whose solution is: $\qquad k = \dfrac{5}{4} \qquad l = -\dfrac{1}{2} \qquad m = \dfrac{1}{4}$

Taking the values $k = 5$, $l = -2$ $m = 1$, we find the matrix:

$$\begin{pmatrix} -5 & 8 & -3 \\ 5 & -4 & 3 \\ 5 & -6 & 1 \end{pmatrix} \text{ whose inverse matrix is } \frac{1}{20}\begin{pmatrix} 7 & 5 & 6 \\ 5 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix}$$

The projectivity which maps the base quadrilateral to the final quadrilateral is:

$$\begin{pmatrix} -6 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -4k & -5\,l & -3\,m \\ 4k & 6\,l & 4\,m \\ k & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

whose solution is: $k = 1$, $l = 1$, $m = -1$ and the square matrix:

$$\begin{pmatrix} -4 & -5 & 3 \\ 4 & 6 & -4 \\ 1 & 0 & 0 \end{pmatrix}$$

Then, the projectivity which maps the initial quadrilateral to the final one is the product:

$$\begin{pmatrix} -4 & -5 & 3 \\ 4 & 6 & -4 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 7 & 5 & 6 \\ 5 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix} = \begin{pmatrix} -68 & -30 & -54 \\ 78 & 30 & 64 \\ 7 & 5 & 6 \end{pmatrix}$$

For example, the point (2, 5/3) is transformed into:

$$\begin{pmatrix} -68 & -30 & -54 \\ 78 & 30 & 64 \\ 7 & 5 & 6 \end{pmatrix} \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 94 \\ -124 \\ 4 \end{pmatrix} = \left( \frac{62}{13}, -\frac{2}{13} \right)$$

A projectivity may be also defined in an alternative way: let $ABCD$ be a quadrilateral and $A'B'C'D'$ another quadrilateral. Then the mapping:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

is a projectivity if the coefficients *a', b', c'* are homogeneous linear functions of *a*, *b*, *c*. For example in the foregoing example the quadrilateral (1, 1)-(2, 3)-(3, 1)-(1, 0) is mapped to the quadrilateral (4, 1)-(6, 0)-(4, 0)-(6, 1). Then we have:

$$(1,0) = (1-b-c)\,(1,1) + b\,(2,3) + c\,(3,1) \quad \Rightarrow \quad b = -\frac{1}{2} \quad c = \frac{1}{4} \quad a = 1-b-c = \frac{5}{4}$$

$$(6,1) = (1-b'-c')\,(4,1) + b'\,(6,0) + c'\,(4,0) \;\Rightarrow\; b' = 1 \quad c' = -1 \quad a' = 1-b'-c' = 1$$

Taking into account that for $a = 1$, $b = 0$, $c = 0 \rightarrow a' = 1$, $b' = 0$, $c' = 0$ and analogously for each of the first three points, the linear mapping of coefficients is a diagonal matrix which can be written simply:

$$a' = \frac{4}{5}a \qquad\qquad b' = -2\,b \qquad\qquad c' = -4\,c$$

If we wish to calculate the transformed point of (2, 5/3) we calculate firstly *a*, *b* and *c*:

$$P = \left(2, \frac{5}{3}\right) = (1-b-c)(1,1) + b\,(2,3) + c\,(3,1) \qquad \Rightarrow \qquad b = \frac{1}{3} \quad c = \frac{1}{3} \quad a = \frac{1}{3}$$

$$\Rightarrow \qquad a' = \frac{4}{15} \qquad b' = -\frac{2}{3} \qquad c' = -\frac{4}{3}$$

Normalising the coefficients, we find the transformed point *P'*:

$$P' = -\frac{4}{26}(4,1) + \frac{10}{26}(6,0) + \frac{20}{26}(4,0) = \left(\frac{62}{13}, -\frac{2}{13}\right)$$

This example shows the normalisation. In general, we say that the following transformation is a projectivity:

$$D = (1-x-y)\,O + x\,P + y\,Q \quad \rightarrow \quad D' = (1-x'-y')\,O' + x'\,P' + y'\,Q'$$

$$x' = \frac{l\,x}{k(1-x-y) + l\,x + m\,y} \qquad\qquad y' = \frac{m\,y}{k\,(1-x-y) + l\,x + m\,y}$$

Let us prove that a projectivity defined in this algebraic way leaves invariant the projective cross ratio (as evident by construction from perspectivities). Under a projectivity, four collinear points *A*, *B*, *C* and *D* are transformed into another four collinear points *A'*, *B'*, *C'* and *D'* . Their cross ratio is the quotient of differences of any coordinate (it is not needed that *x* and *y* are Cartesian coordinates) because these differences for *x* are proportional to the differences for *y* due to the alignment:

$$x'_C - x'_A = \frac{l\left[k\left(x_C - x_A\right) + \left(k - m\right)\left(x_A y_C - x_C y_A\right)\right]}{\left[k\left(1 - x_C - y_C\right) + l\, x_C + m\, y_C\right]\left[k\left(1 - x_A - y_A\right) + l\, x_A + m\, y_A\right]}$$

Let us suppose that the collinear points $A$, $B$, $C$ and $D$ lie on the line having the equation $y = s\,x + t$, where the constants $s$ and $t$ play the role of a "slope" and "ordinate intercept" although the $x$-axis and the $y$-axis be not orthogonal:

$$x'_C - x'_A = \frac{l\left(x_C - x_A\right)\left[k - \left(k - m\right)t\right]}{\left[k\left(1 - x_C - y_C\right) + l\, x_C + m\, y_C\right]\left[k\left(1 - x_A - y_A\right) + l\, x_A + m\, y_A\right]}$$

Now we see that a projectivity changes every single ratio due to the denominators, but the cross ratio is preserved because these denominators are cancelled in this case:

$$\left(A'\,B'\,C'\,D'\right) = \frac{\left(x_C{}' - x_A{}'\right)\,\left(x_D{}' - x_B{}'\right)}{\left(x_D{}' - x_A{}'\right)\,\left(x_C{}' - x_B{}'\right)} = \frac{\left(x_C - x_A\right)\,\left(x_D - x_B\right)}{\left(x_D - x_A\right)\,\left(x_C - x_B\right)} = \left(A\,B\,C\,D\right)$$

### The projectivity as a tool for theorems demonstration

One of the most interesting applications of the projectivity is the demonstration of theorems of projective nature owing to the fact that the projectivity preserves alignment and incidence. For example, in the figure 10.3 I have displayed without proof the method of construction of the fourth point which completes a harmonic range from the other three. Now let us apply to this figure a projectivity which maps the point $A$ to the infinity. In this case the lines $ABCD$, $R$ and $S$ become parallel (figure 10.7) and the cross ratio becomes a single ratio:

$$(A\,B\,C\,D) = (\infty\, B'\,C'\,D') = \frac{B'D'}{B'C'} = 2$$

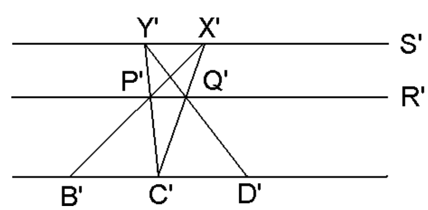
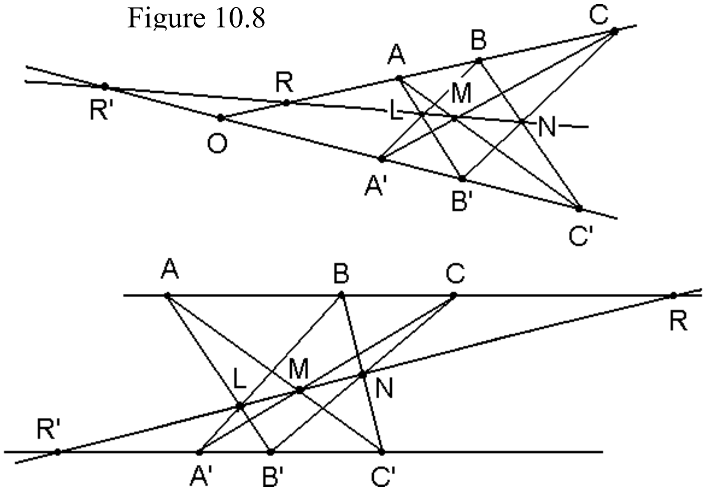



Figure 10.7

By similarity of triangles one sees that $B'D' = 2\, B'C'$, so that the cross ratio is 2 and proves that the points $A$, $B$, $C$ and $D$ in the figure 10.3 are a harmonic range.

Another example is the proof of the Pappus' theorem: the lines joining points with distinct letters belonging to two distinct ranges $ABC$ and $A'B'C'$ intersect in three collinear points $L$, $M$ and $N$ (figure 10.8). Applying a perspectivity to make both ranges parallel, now we see that the triangles $ABL$ and $A'B'L$ are

Figure 10.8



similar so that:

$$L = \frac{|A'B'|}{|AB| + |A'B'|} A + \frac{|AB|}{|AB| + |A'B'|} B' \quad \text{and also} \quad L = \frac{|A'B'|}{|AB| + |A'B'|} B + \frac{|AB|}{|AB| + |A'B'|} A'$$

We may gather both equations obtaining an expression more adequate to our purposes which is function of the midpoints:

$$L = \frac{|A'B'|}{|AB| + |A'B'|} \frac{A+B}{2} + \frac{|AB|}{|AB| + |A'B'|} \frac{A'+B'}{2}$$

Analogously for $M$ and $N$ we obtain:

$$M = \frac{|A'C'|}{|AC| + |A'C'|} \frac{A+C}{2} + \frac{|AC|}{|AC| + |A'C'|} \frac{A'+C'}{2}$$

$$N = \frac{|B'C'|}{|BC| + |B'C'|} \frac{B+C}{2} + \frac{|BC|}{|BC| + |B'C'|} \frac{B'+C'}{2}$$

These expressions fulfil the following equality what implies that the three points are aligned:

$$M = \frac{|AB| + |A'B'|}{|AC| + |A'C'|} L + \frac{|BC| + |B'C'|}{|AC| + |A'C'|} N$$

To prove this, begin from

$$\left(|AC| + |A'C'|\right) M = \left(|AB| + |A'B'|\right) L + \left(|BC| + |B'C'|\right) N \quad \Rightarrow$$

$$|A'C'|\left(A+C\right) + |AC|\left(A'+C'\right) = |A'B'|\left(A+B\right) + |AB|\left(A'+B'\right) + |B'C'|\left(B+C\right) + |BC|\left(B'+C'\right)$$

Arranging $A$, $B$ and $C$ on the left hand side and $A'$, $B'$ and $C'$ on the right hand side and taking into account that $|AC| = |AB| + |BC|$ and $|A'C'| = |A'B'| + |B'C'|$ we obtain the following vector equality:

$$-|B'C'|\, AB + |A'B'|\, BC = -|AB|\, B'C' + |BC|\, A'B'$$

which is an identity taking into account that the lines $ABC$ and $A'B'C'$ are parallel and all vectors have the same direction and sense.

The linear relation between $L$, $M$ and $N$ is very useful to calculate the cross ratio ($LMNR$). In fact the coefficients of the midpoints are the relative altitudes of the point with respect to both parallel lines. Since this cross ratio is also the quotient of the relative altitudes, we have:

$$\left(LMNR\right)=\frac{\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'B'|}{|AB|+|A'B'|}\right)\dfrac{-|AC|}{|AC|+|A'C'|}}{\dfrac{-|AB|}{|AB|+|A'B'|}\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'C'|}{|AC|+|A'C'|}\right)}=\frac{|AC|}{|AB|}=(OABC)$$

The single ratio is the cross ratio corresponding to $O=\infty$. This identity of cross ratios is preserved under a projectivity and it is also valid for the upper scheme on the figure 10.8 when $O$ is a point with finite coordinates. Analogously:

$$\left(LMNR'\right)=\frac{\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'B'|}{|AB|+|A'B'|}\right)\dfrac{|A'C'|}{|AC|+|A'C'|}}{\dfrac{|A'B'|}{|AB|+|A'B'|}\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'C'|}{|AC|+|A'C'|}\right)}=\frac{|A'C'|}{|A'B'|}=(OA'B'C')$$

### The homology

The points $A'$, $B'$ are *homologous* of the points $A$ and $B$ with respect to the *centre* $O$ and the *axis of homology* given by the point $F$ and the vector $v$ (figure 10.9) if:

1) The centre $O$ does not lie on the axis.

2) Every pair of homologous points and the homology centre are aligned, that is, $O$, $A$ and $A'$ are aligned and also $O$, $B$ and $B'$.

3) Every pair of homologous lines $AB$ and $A'B'$ intersects in a point on the homology axis.

A homology[6] is determined by an axis, a centre $O$ and a pair of homologous points $A$ and $A'$. To obtain the homologous of any point $B$, we draw the line $AB$, whose intersection with the homology axis will be denoted as $Z$. Then we draw the line $A'Z$ and the intersection with the line $OB$ is the homologous point $B'$. By construction, the homology axis and the lines $AB$, $A'B'$ and $OZ$ are concurrent in $Z$. The projective cross ratio of this pencil of lines is independent of the crossing line where it is measured. Let $G$ and $H$ be the intersections of the homology axis with the lines $OA$ and $OB$ respectively. Then we have:

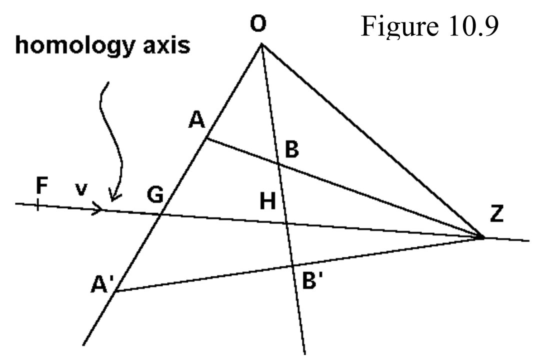



Figure 10.9

$$\{ Z, O\,A\,G\,A' \} = \{ Z, O\,B\,H\,B' \}$$

[6] The word *homology* is used by H. Eves (*A Survey of Geometry,* p. 105) with a different meaning. He names as *homology* a composite homothety, that is, a single homothety followed by a rotation.

Since the four points of each set are collinear, their projective cross ratio is identical to their complex cross ratio:

$$( O\,A\,G\,A' ) = ( O\,B\,H\,B' )$$

That is, the cross ratio of the four points $O$, $B$, $H$ and $B'$ is a real constant $r$ for any point $B$ and is called *homology ratio*. Hence we can write:

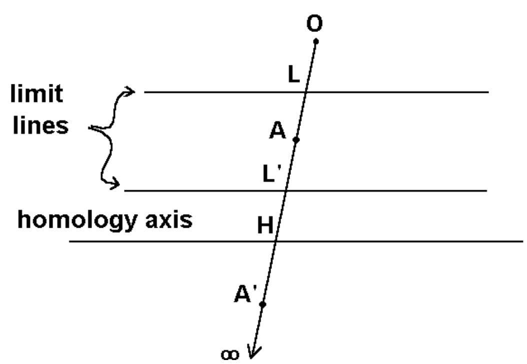
$$( O\,B\,H\,B' ) = OH\,OB'^{-1}\,BB'\,BH^{-1} = r \neq 1, 0$$

This equation leads to a simpler definition of the homology: $B'$ is the homologous point of $B$ with respect a given axis and centre $O$ (outside the axis) with ratio $r$ if $O$, $B$ and $B'$ are aligned and the cross ratio $( O\,B\,H\,B' )$ is equal to $r$, being $H$ the intersection point of the line $OB$ with the homology axis. Hence, a homology is determined by giving a centre $O$, an axis and the homology ratio. If the homology ratio is the unity we have two degenerate cases: whether $B$ lies on $O$ and $B'$ is any point, or $B'$ lies on $H$ (on the homology axis) and $B$ is any point, that is, there is not one to one mapping, a not useful transformation. Then we must impose a ratio distinct of the unity. On the other hand, a null homology ratio implies that $BB' = 0$ and the homology becomes the identity.

In the special case $r = 2$ (when the points $A$ and $A'$ are harmonic conjugates with respect $O$ and $H$) the transformation is called a *harmonic homology*.

The *first limit line* is defined as the set of points $L$ on the plane having as homologous points those at infinity (figure 10.10):

Figure 10.10




$$( O\,L\,H\,\infty ) = OH\,O\infty^{-1}\,L\infty\,LH^{-1} = r$$

The limit of the quotient of both distances tending to the infinity is the unity:

$$O\infty^{-1}\,L\infty = \lim_{X \to \infty} OX^{-1}\,LX = 1$$

Then: $\qquad OH\,LH^{-1} = r \qquad \Rightarrow \qquad LH\,OH^{-1} = r^{-1}$

Because all the points $H$ lie on the homology axis, all the points $L$ form a line, the limit line.

The *second limit line* is defined as the points that are homologous of those at infinity:

$$( O\,\infty\,H\,L' ) = OH\,OL'^{-1}\,\infty L'\,\infty H^{-1} = r$$

$$OH\,OL'^{-1} = r \qquad \Rightarrow \qquad OL'\,OH^{-1} = \mathrm{r}^{-1}$$

Therefore the distance from the second limit line to the homology axis is equal to the distance from the first limit line to the homology centre (figure 10.10), and both lines are located whether between or outside both elements according to the value of $r$:

$$OL' = LH = r^{-1}\,OH$$

Let us calculate the homologous point $B'$ for every point $B$ on the plane. Since $H$ is a changing point on the homology axis, we write it as a function of a fixed point $F$ also on the axis and its direction vector $v$ (figure 10.9). At one time, the point $H$ is the intersection of the line $OB$ and the homology axis:

$$H = O + b\,OB = F + k\,v \qquad\qquad k,\, b \text{ real}$$

$$b\,OB - k\,v = OF \qquad \Rightarrow \qquad b = OF \wedge v\,(\,OB \wedge v\,)^{-1}$$

$$H = O + OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

Now we may calculate the segments $OH$ and $BH$:

$$OH = OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

$$BH = BO + OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB = [\,OF \wedge v\,(\,OB \wedge v\,)^{-1} - 1\,]\,OB =$$

$$= (\,OF \wedge v - OB \wedge v\,)\,(\,OB \wedge v\,)^{-1}\,OB = BF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

By substitution in the cross ratio we obtain:

$$OH\,OB'^{-1}\,BB'\,BH^{-1} = OH\,BH^{-1}\,BB'\,OB'^{-1} = OF \wedge v\,(\,BF \wedge v\,)^{-1}\,BB'\,OB'^{-1} = r$$

$$BB'\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}$$

and isolating $OB'$ as a function of $OB$ and other known data:

$$(\,-OB + OB'\,)\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1} - OB\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1} - 1$$

$$OB'^{-1} = OB^{-1}\,[\,1 - r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}\,] = OB^{-1}\,(\,v \wedge OF - r\,v \wedge BF\,)\,(\,v \wedge OF\,)^{-1}$$

Inverting the vectors we find:

$$OB' = OB\,v \wedge OF\,(\,v \wedge OF - r\,v \wedge BF\,)^{-1} = OB\,(\,1 - r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}\,)^{-1}$$

This equation allows to calculate the homologous point $B'$ of any point $B$ using the data of the homology: the centre $O$, the ratio $r$, and a direction vector $v$ and a point $F$ of the axis.

The homology preserves the projective cross ratio of any set of points. Let $A'$, $B'$, $C'$, $D'$ and $E'$ be the transformed points of $A$, $B$, $C$, $D$ and $E$ under the homology. Then:

$$\{ E, A\, B\, C\, D \} = \{ E', A'\, B'\, C'\, D' \}$$

Let us prove this statement. Being $A'$ and $E'$ homologous points of $A$ and $E$ we have:

$$OA' = OA\,(\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,)^{-1}$$

$$OE' = OE\,(\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,)^{-1}$$

Thus the segment $E'A'$ is:

$$E'A' = OA' - OE'$$

$$E'A' = OA\,(\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,)^{-1} - OE\,(\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,)^{-1} =$$

$$= [\,OA\,(\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,) - OE\,(\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,)\,]$$

$$[\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\,EA + r\,(\,-OA\,v \wedge EF + OE\,v \wedge AF\,)\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\,EA\,(\,1 - r\,) + r\,(\,OA\,v \wedge OE - OE\,v \wedge OA\,)\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\,EA\,(\,1 - r\,) + r\,v\,OA \wedge OE\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge AF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1}$$

For the other vectors analogous expressions are obtained:

$$E'B' = [\,EB\,(\,1 - r\,) + r\,v\,OB \wedge OE\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1}$$

$$E'C' = [\,EC\,(\,1 - r\,) + r\,v\,OC \wedge OE\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge CF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1}$$

$$E'D' = [\,ED\,(\,1 - r\,) + r\,v\,OD \wedge OE\,(\,v \wedge OF\,)^{-1}\,]$$

$$[\,1 - r\,v \wedge DF\,(\,v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\,v \wedge EF\,(\,v \wedge OF\,)^{-1}\,]^{-1}$$

Let us calculate the outer product $E'A' \wedge E'C'$:

$$E'A' \wedge E'C' = [\, EA \wedge EC\, (\, 1 - \mathrm{r}\, )^2 + (\, EA \wedge v\, OC \wedge OE + v \wedge EC\, OA \wedge OE\, )\, r\, (\, 1 - r\, )$$

$$(\, v \wedge OF\, )^{-1}\, ]\, [\, 1 - r\, v \wedge AF\, (\, v \wedge OF\, )^{-1}\, ]^{-1}\, [\, 1 - r\, v \wedge CF\, (\, v \wedge OF\, )^{-1}\, ]^{-1}$$

$$[\, 1 - r\, v \wedge EF\, (\, v \wedge OF\, )^{-1}\, ]^{-2}$$

By substitution of the following identity, which may be proved by means of the geometric algebra (see exercise 1.4):

$$EA \wedge v\, OC \wedge OE + v \wedge EC\, OA \wedge OE = EA \wedge v\, EC \wedge OE + v \wedge EC\, EA \wedge OE =$$

$$EA \wedge EC \;\; v \wedge OE$$

$$E'A' \wedge E'C' = [\, EA \wedge EC\, (\, 1 - r\, )^2 + EA \wedge EC\, v \wedge OE\, r\, (\, 1 - r\, )\, (\, v \wedge OF\, )^{-1}\, ]$$

$$[1 - r\, v \wedge AF\, (v \wedge OF\, )^{-1}\, ]^{-1}\, [1 - r\, v \wedge CF\, (\, v \wedge OF\, )^{-1}\, ]^{-1}\, [1 - r\, v \wedge EF\, (\, v \wedge OF\, )^{-1}\, ]^{-2}$$

Extracting common factor:

$$E'A' \wedge E'C' = EA \wedge EC\, [\, (\, 1 - r\, )^2 + v \wedge OE\, r\, (\, 1 - r\, )\, (\, v \wedge OF\, )^{-1}\, ]$$

$$[1 - r\, v \wedge AF\, (v \wedge OF\, )^{-1}\, ]^{-1}\, [1 - r\, v \wedge CF\, (\, v \wedge OF\, )^{-1}\, ]^{-1}\, [1 - r\, v \wedge EF\, (\, v \wedge OF\, )^{-1}\, ]^{-2}$$

In the projective cross ratio, all the factors are simplified except the first outer product:

$$\frac{E'A' \wedge E'C'\; E'B' \wedge E'D'}{E'A' \wedge E'D'\; E'B' \wedge E'C'} = \frac{EA \wedge EC\; EB \wedge ED}{EA \wedge ED\; EB \wedge EC}$$

This proves that the projective cross ratio of any four points $A$, $B$, $C$ and $D$ with respect to a centre of projection $E$ is equal to the projective cross ratio of the homologous points $A'$, $B'$, $C'$ and $D'$ with respect to the homologous centre of projection $E'$. It means that the homology is a special kind of projectivity where a line is preserved, the axis of homology. When the centre of projection is a point $H$ on the homology axis, $H$ and $H'$ are coincident and we find in the former equality the initial condition of the homology again:

$$H = H' \qquad \{\, H, A'\, B'\, C'\, D'\, \} = \{\, H, A\, B\, C\, D\, \}$$

The following chapter is devoted to the conics and the Chasles' theorem, which states that the locus of the points from where the projective cross ratio of any four points is constant is a conic passing through these four points. Since the homology preserves the projective cross ratio, it transform conics into conics. Hence, a conic may be drawn as the homologous curve of a circle. Depending on the position of this circle there are three cases:

1) The circle does not cut the limit line $L$ (whose homologous is the line at infinity). In this case the homologous curve is an ellipse, because very next points on the circle have also very next homologous points and therefore the homologous curve must be closed.

2) The circle touches the limit line $L$. Then its homologous curve is the parabola, because the homologous of the contact point is a point at infinity. The symmetry axis of

the parabola has the direction of the line passing through the centre of homology and the contact point.

3) The circle cuts the limit line $L$. Then the homologous curve is the hyperbola because the homologous points of both intersections are two points at infinity. The lines passing through the centre of homology and both intersection points have the direction of the asymptotes.

### Exercises

10.1 Prove the Ptolemy's theorem: For a quadrilateral inscribed in a circle, the product of the lengths of both diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

10.2 Find that the point $D$ forming a harmonic range with $A$, $B$ and $C$ is given by the equation:

$$D = A + AB\,(\,1 - 2\,AC^{-1}\,BC\,)^{-1}$$

10.3 Show that the homography is a directly conformal transformation, that is, it preserves angles and their orientations.

10.4 Prove that if $A'$, $B'$, $C'$ and $D'$ are the homologous of $A$, $B$, $C$ and $D$, and $H$ is a point on the homology axis, then the equality $\{\,H, A'\,B'\,C'\,D'\,\} = \{\,H, A\,B\,C\,D\,\}$ holds. That is, show directly that:

$$\frac{HA' \wedge HC'\;HB' \wedge HD'}{HA' \wedge HD'\;HB' \wedge HC'} = \frac{HA \wedge HC\;HB \wedge HD}{HA \wedge HD\;HB \wedge HC}$$

10.5 A *special conformal transformation* of centre $O$ and translation $v$ is defined as the geometric operation which, given any point $P$, consists in the inversion of the vector $OP$, the addition of the translation $v$ and the inversion of the resulting vector again.

$$OP'^{-1} = (\,OP^{-1} + v\,)^{-1}$$

Prove the following properties of this transformation:

a) It is an additive operation with respect to the translations: the result of applying firstly a transformation with centre $O$ and translation $v$, and later a transformation with the same centre and translation $w$ is identical to the result of applying a unique transformation with translation $v + w$.
b) It preserves the complex cross ratio and thus it is a special case of homography.

10.6 Prove that if a homography keeps invariant three or more points on the plane, then it is the identity.

10.7 An *antigraphy* is defined as that transformation which conjugates the complex cross ratio of any four points:

$$( A\ B\ C\ D ) = ( A'\ B'\ C'\ D' )^*$$

Then an antigraphy is completely specified by giving the images $A'$, $B'$, $C'$ of any three points $A$, $B$, $C$. Prove that:

a) The antigraphy is an opposite conformal transformation, that is, it changes the orientation of the angles but preserving their absolute values.
b) The composition of two antigraphies is a homography.
c) A product of three inversions is an antigraphy.
d) If an antigraphy has three invariant points, then it is a circular inversion.

10.8 Let $A$, $B$, $C$ and $A'$, $B'$, $C'$ be two sets of independent points. A projectivity has been defined as the mapping:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

where the coefficients $a'$, $b'$, $c'$ are homogeneous linear functions of $a$, $b$, $c$. Prove that collinear points are mapped to collinear points.

10.9 Prove the following theorem: let the sides of a hexagon *ABCDEF* pass alternatively through the points $P$ and $Q$. Then the lines *AD*, *BE* and *CF* joining opposite vertices meet in a unique point. Hint: draw the hexagon in the dual plane.

## 11. CONICS

### Conic sections

The *conic sections* (or *conics*) are the intersections of a conic surface with any transversal plane (figure 11.1). The *proper conics* are the curves obtained with a plane not passing through the vertex of the cone. If the plane contains the vertex we have improper conics, which reduce to a pair of straight lines or a point. In general, I shall regard only proper conics, taking into account that the other case can be usually obtained as a limit case.

Let us consider the two spheres inscribed in the cone which are tangent to the

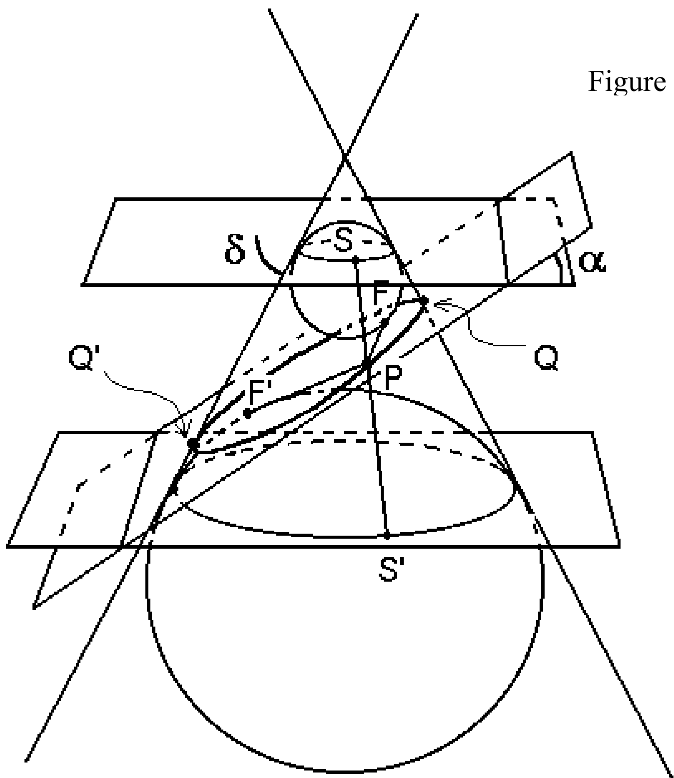



Figure 11.1

plane of the conic. The spheres touch this plane at the points $F$ and $F'$, the *foci* of the conic section. On the other hand, both spheres touch the cone surface at two tangency circles lying in planes perpendicular to the cone axis. The *directrices* of the conic are defined as the intersections of these planes with the plane of the conic.

Let $P$ be a point on the conic. Since it lies on the plane of the conic, the segment going from $P$ to the focus $F$ is tangent to the sphere. Since it belongs to the cone surface, the generatrix passing through $P$ is also tangent to the sphere. $PF$ and $PS$ are tangent to the same sphere so that their lengths are equal: $|PF|=|PS|$. Also $PF'$ and $PS'$ are tangent to the other sphere so that their lengths are also equal: $|PF'|=|PS'|$. On the other hand the segment $SS'$ of any directrix has constant length, what implies that the addition of distances from any point $P$ on the conic to both foci is constant:

$$|PF| + |PF'| = |PS| + |PS'| = |SS'| = \text{constant}$$

This discussion slightly changes for a plane which intercepts the upper and lower cones. In this case, the distances from $P$ to the upper sphere and to the upper branch of the curve must be considered negative[1]. If the plane is parallel to a generatrix of the cone these distances become infinite. This unified point of view for all conics using negative or infinite distances when needed will be the guide of this chapter. I shall not separate equations but only specify cases for distinct conics.

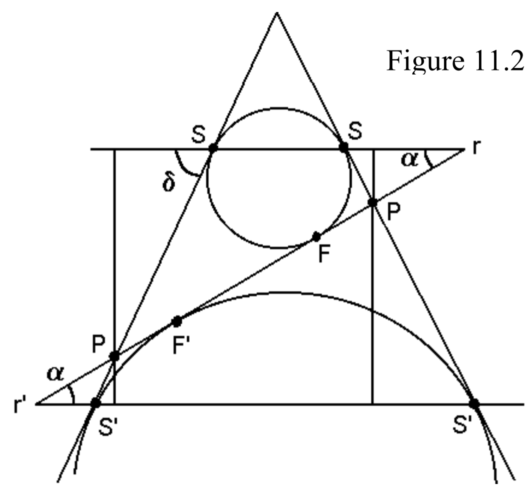
A lateral view of figure 11.1 is given in the figure 11.2. In this figure the point $P$ has been drawn in both extremes of the conic (and $S$ and $S'$ also twice), although we must consider any point on the conic section. As indicated previously $|PF| = |PS|$. But $|PS|$ is proportional to the distance from $P$ to the upper plane containing $S$ and this distance is also proportional to the distance from $P$ to the directrix $r$ (figure 11.2):

$$|PF| = |PS| = \frac{\sin\alpha}{\sin\delta}\, \mathrm{d}(P, r)$$

The quotient of the sines of both angles is called the *eccentricity* e of the conic:

$$\mathrm{e} = \frac{\sin\alpha}{\sin\delta} > 0$$

Figure 11.2




Then a conic can be also defined as the geometric locus of the points $P$ whose distance from the focus $F$ is proportional to the distance from the directrix $r$ (figure 11.3).

$$|FP| = \mathrm{e}\, \mathrm{d}(P, r)$$

$|FP|$ is called the *focal radius*. Let $T$ be the point of the directrix closest to the focus (figure 11.3). Then the *main axis of symmetry* of the conic is the line which is perpendicular to the directrix and passes through the focus. Under a reflection with respect the main axis of symmetry, the conic is preserved. The oriented distance from any point $P$ to the directrix $r$ is expressed, using the vector $FT$, as:

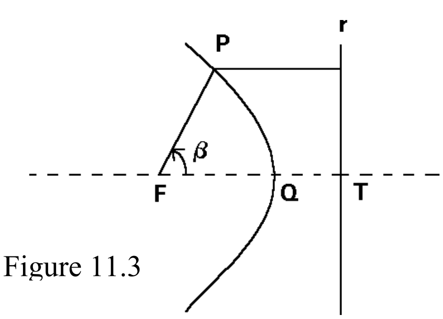



Figure 11.3

[1] To consider sensed distances is not a trouble but an advantage. For example, the distance from a point to a line can be also taken as an oriented distance, whose sign indicates the half-plane where the point lies.

$$\mathrm{d}(P, r) = \frac{FT \cdot PT}{|FT|}$$

Defined in this way, the distance from $P$ to $r$ is positive when $P$ is placed on the half-plane that contains the focus of both in which the directrix divides the plane, and negative when $P$ lies on the other half-plane. The equation of the conic becomes:

$$|FP| = \mathrm{e}\,\mathrm{d}(P, r) = \mathrm{e}\frac{FT \cdot PT}{|FT|}$$

$$|FP|\;|FT| = \mathrm{e}\,FT \cdot PT = \mathrm{e}\,FT \cdot (FT - FP) = \mathrm{e}\left(FT^2 - FT \cdot FP\right)$$

Let us denote by $\beta$ the angle between the main axis $FT$ and the focal vector $FP$:

$$|FP|\;|FT| = \mathrm{e}\left(FT^2 - |FT|\;|FP|\cos\beta\right)$$

$$|FP|\;|FT|\,(1 + \mathrm{e}\;\cos\beta) = \mathrm{e}\,FT^2$$

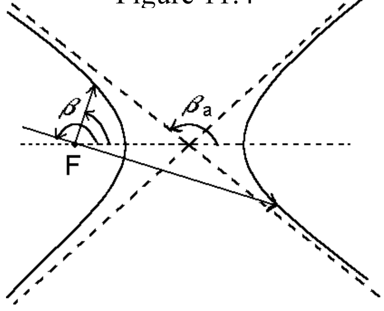
From here one obtains the *polar equation* of a conic:

$$|FP| = \frac{|FT|}{1 + \mathrm{e}\;\cos\beta}$$

According to the value of the eccentricity there are three types of proper conics:

1) For $0 < \mathrm{e} < 1$, the inclination $\alpha$ of the plane of the conic is lower than the inclination $\delta$ of the cone generatrix and the denominator is always positive. Then the focal radius is always positive and the points $P$ form a closed curve on the focal half-plane, an *ellipse*.
2) For $\mathrm{e} = 1$, $\alpha = \delta$, the plane of the conic is parallel to a cone generatrix. Since the denominator vanishes for $\beta = \pi$, the points $P$ form an open curve called *parabola*. Except for this value of $\beta$, the radius is always positive and the curve lies on the focal half-plane.
3) For $\mathrm{e} > 1$, $\alpha > \delta$, the plane of the conic intercepts both upper and lower cones. The denominator vanishes twice so that it is an open curve with two branches called *hyperbola* (figure 11.4). The lowest positive angle that makes the denominator zero is the asymptote angle $\beta_a$ :

Figure 11.4




$$\beta_a = \arccos\left(-\frac{1}{\mathrm{e}}\right)$$

The maximum eccentricity of the hyperbolas as cone sections is obtained with $\alpha = \pi/2$:

$$1 < \mathrm{e} \le \frac{1}{\sin\delta}$$

The focal radius is positive ($P$ lies on the focal branch) for the ranges of $\beta$:

$$0 < \beta < \beta_a \qquad \text{and} \qquad 2\pi - \beta_a < \beta < 2\pi$$

and negative ($P$ lies on the non focal branch) for the range:

$$\beta_a < \beta < 2\pi - \beta_a$$

A negative focal radius means that we must take the sense opposite that determined by the angle (figure 11.4). For example, a radius –2 with an angle $7\pi/6$ is equivalent to a radius 2 with an angle of $\pi/6$. So the vertex $T'$ of the non focal branch has an angle $\pi$ (not zero as it would appear) and a negative focal radius.

**Two foci and two directrices**

Figure 11.5

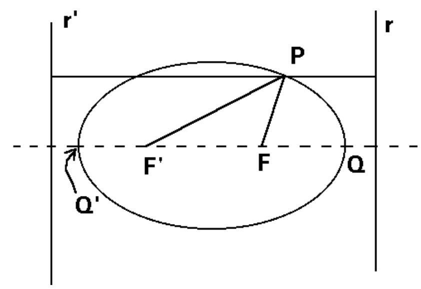
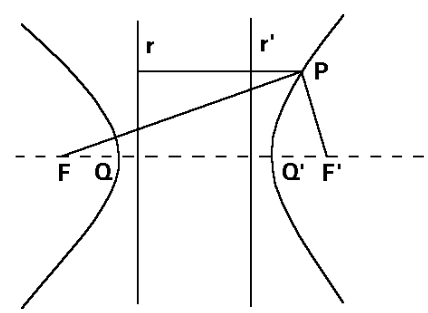



Figure 11.6




There are usually two spheres touching the cone surface and the conic plane, and hence there are also two foci. For the case of the ellipse, both tangent spheres are located in the same cone. For the case of the hyperbola, each sphere is placed in each cone so that each focus is placed next to each branch. In the case of the parabola, one sphere and the corresponding focus is placed at the infinity. For the improper conics, the tangent spheres have null radius and the foci are coincident with the vertex of the cone, which is the crossing point of both lines.

As the figure 11.1 shows for a proper conic, both directrices $r$ and $r'$ are parallel, and both foci $F$ and $F'$ are located on the axis of symmetry of the conic. Above we have already seen that the addition of oriented distances from both foci to any point $P$ is constant. Let us see the relation with the distance between the directrices. If $P$ is any point on the conic then:

$$|FP| + |F'P| = \mathrm{e}\,\mathrm{d}(P,r) + \mathrm{e}\,\mathrm{d}(P,r') = \mathrm{e}\,\mathrm{d}(r,r')$$

The sum of the oriented distances from any point $P$ to any two parallel lines (the directrices in our case) is constant[2]. Then the addition of both focal radii of any point $P$

[2] Note that this statement is only right for oriented distances but not for positive distances.

on the conic is the product of the eccentricity multiplied by the oriented distance between both directrices. *P* can be any point, for example *Q*, the extreme of the conic:

$$|FP|+|F'P|=|FQ|+|F'Q|=|F'Q'|+|F'Q|=|QQ'|$$

whence it follows that the addition of both focal radii is $|QQ'|$, the *major diameter*. Specifying:

1) For the case of the ellipse (figure 11.5), both focal radii are positive and also the major diameter.
2) For the case of the parabola, one directrix is the line at the infinity and one focus is a point at the infinity on the main axis of symmetry. Hence the major diameter has an infinite value.
3) For the case of the hyperbola (figure 11.6), the focal radius of a point on the non focal branch is negative, its absolute value being higher than the other focal radius, which is positive, yielding a negative major diameter.

**Vectorial equation**

The polar equation of a conic may be written having as parameter the *focal distance* $|FQ|$ instead of the distance from the focus to the directrix $|FT|$:

$$|FP|=\frac{1+\mathrm{e}}{1+\mathrm{e}\ \cos\beta}|FQ|$$

Let $Q'$ be the vertex of the conic closest to the focus $F'$. The distance from the focus $F$ to $Q'$ is found for $\beta=\pi$:

$$|FQ'|=\frac{1+\mathrm{e}}{1-\mathrm{e}}|FQ|$$

For the ellipse $|FQ'|$ is a positive distance; for the hyperbola it is a negative distance and for the parabola $Q'$ is at infinity. Then the distance between both foci is:

$$|FF'|=|FQ'|-|F'Q'|=|FQ'|-|FQ|=\frac{2\ \mathrm{e}}{1-\mathrm{e}}|FQ|$$

On the other hand the major diameter is:

$$|QQ'|=|FQ|+|FQ'|=\frac{2|FQ|}{1-\mathrm{e}}$$

By dividing both equations, an alternative definition of the eccentricity is obtained:

$$\mathrm{e}=\frac{|FF'|}{|QQ'|}$$

The eccentricity is the ratio of the distance between both foci divided by the major diameter. For the case of ellipse, both distances are positive. For the case of the hyperbola, both distances are negative, so the eccentricity is always positive. When e = 0 both foci are coincident in the centre of a circle. Note that the circle is obtained when we cut the cone with a horizontal plane. In this case, the directrices are the line at the infinity.

The *vectorial equation* of a conic is obtained from the polar equation and contains the radius vector $FP$. Since $FP$ forms with $FQ$ an angle $\beta$ (figure 11.3), $FP$ is obtained from the unitary vector of $FQ$ via multiplication by the exponential with argument $\beta$ and by the modulus of $FP$ yielding:

$$FP = \frac{1+\mathrm{e}}{1+\mathrm{e}\cos\beta} FQ \left(\cos\beta + e_{12}\sin\beta\right)$$

On the other hand, from the directrix property, one easily finds the following equation for a conic:

$$FP^2\, FT^2 = \mathrm{e}^2\,(\, FT^2 - FT\cdot FP\,)^2$$

$F$, $T$ and e are parameters of the conic, and $P = (x, y)$ is the mobile point. Therefore from this equation we will also obtain a Cartesian equation of second degree. For example, let us calculate the Cartesian equation of an ellipse with eccentricity ½ and focus at the point (3, 4) and vertex at (4,5):

$$\mathrm{e} = 1/2 \qquad F = (\,3, 4\,) \qquad Q = (\,4, 5\,) \qquad P = (\,x, y\,)$$

$$FT = \frac{1+\mathrm{e}}{\mathrm{e}} FQ = 3e_1 + 3e_2 \qquad T = F + FT = (\,6, 7\,)$$

$$FP = (\,x-3\,)\,e_1 + (\,y-4\,)\,e_2$$

Then the equation of the conic is:

$$[\,(\,x-3\,)^2 + (\,y-4\,)^2\,]\;18 = \frac{1}{4}\,[\,18 - (\,3\,(x-3) + 3\,(\,y-4\,)\,)\,]^2$$

After simplification:

$$7\,x^2 - 2\,x\,y + 7\,y^2 - 22\,x - 38\,y + 31 = 0$$

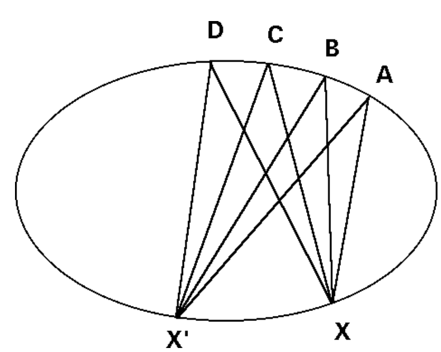


Figure 11.7

### The Chasles' theorem

According to this theorem[3], the projective cross ratio of any four given points $A$, $B$, $C$ and $D$ on a conic with respect to a point $X$ also on this conic is constant independently of the choice of

[3] Michel Chasles, *Traité des sections coniques*, Gauthier-Villars, Paris, 1865, p. 3.

the point $X$ (figure 11.7):

$$\{X, A\,B\,C\,D\} = \{X', A\,B\,C\,D\}$$

To prove this theorem, let us take into account that the points $A$, $B$, $C$, $D$ and $X$ must fulfil the vectorial equation of the conic. Let us also suppose, without loss of generality, the main axis of symmetry having the direction $e_1$ (this supposition simplifies the calculations):

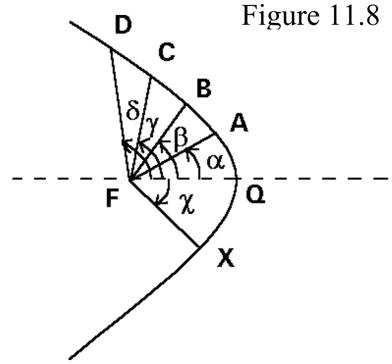



Figure 11.8

$$FQ = |FQ|\; e_1$$

Now on $\alpha$, $\beta$, $\gamma$, $\delta$ and $\chi$ will be the angles which the focal radii $FA$, $FB$, $FC$, $FD$ and $FX$ form with the main axis with direction vector $FQ$ (figure 11.8). Then:

$$XA = FA - FX = |FQ|(1+\mathrm{e})\left[\frac{e_1\cos\alpha + e_2\sin\alpha}{1+\mathrm{e}\cos\alpha} - \frac{e_1\cos\chi + e_2\sin\chi}{1+\mathrm{e}\cos\chi}\right]$$

Introducing a common denominator, we find:

$$XA = |FQ|\frac{(1+\mathrm{e})[e_1(\cos\alpha - \cos\chi) + e_2(\sin\alpha - \sin\chi + \mathrm{e}\sin\alpha\cos\chi - \mathrm{e}\cos\alpha\sin\chi)]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\chi)}$$

From $XA$ and the analogous expression for $XC$, and after simplification we obtain:

$$XA\wedge XC = FQ^2\,\frac{e_{12}(1+\mathrm{e})^2(\sin\gamma\cos\alpha - \sin\alpha\cos\gamma + \sin\chi\cos\gamma - \sin\gamma\cos\chi + \sin\alpha\cos\chi - \sin\chi\cos\alpha)}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}$$

$$= FQ^2\,\frac{(1+\mathrm{e})^2[\sin(\gamma-\alpha) + \sin(\chi-\gamma) + \sin(\alpha - \chi)]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

Using the trigonometric identities of the half-angles, the sum is converted into a product of sines (exercise 6.2):

$$XA\wedge XC = -4\,FQ^2\,\frac{(1+\mathrm{e})^2\left[\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\chi-\gamma}{2}\right)\sin\left(\frac{\alpha-\chi}{2}\right)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

In the same way the other outer products are obtained. The projective cross ratio is their quotient, where the factors containing the eccentricity or the angle $\chi$ are simplified:

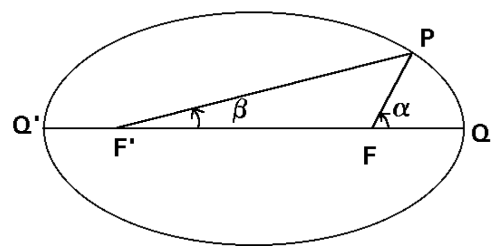
$$\{X, A\,B\,C\,D\} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} = \frac{\sin\dfrac{\gamma-\alpha}{2}\,\sin\dfrac{\delta-\beta}{2}}{\sin\dfrac{\delta-\alpha}{2}\,\sin\dfrac{\gamma-\beta}{2}} = \frac{\sin\dfrac{AFC}{2}\,\sin\dfrac{BFD}{2}}{\sin\dfrac{AFD}{2}\,\sin\dfrac{BFC}{2}}$$

since $\gamma-\alpha$ is the angle *AFC*, etc. Therefore, the projective cross ratio of four points *A*, *B*, *C* and *D* on a conic is equal to the quotient of the sines of the focal half-angles, which do not depend on *X*, but only on the positions of *A*, *B*, *C* and *D*, fact which proves the Chasles' theorem. This statement is trivial for the case of the circumference, because the inscribed angles are the half of the central angles. However the inscribed angles on a conic vary with the position of the point *X* and they differ from the half focal angles. In spite of this, it is a notable result that the quotient of the sines of the inscribed angles (projective cross ratio) is equal to the quotient of the half focal angles. For the case of the hyperbola I remind you that the focal radius of a point on the non focal branch is oriented with the opposite sense and the focal angle is measured with respect this orientation.

### Tangent and perpendicular to a conic

The vectorial equation of a conic with the major diameter oriented in the direction $e_1$ (figure 11.9) is:

Figure 11.9




$$FP = \frac{(1+\mathrm{e})\,|\,FQ\,|}{1+\mathrm{e}\cos\alpha}\left(e_1\cos\alpha + e_2\sin\alpha\right)$$

The derivation with respect the angle $\alpha$ gives:

$$\frac{d\,FP}{d\alpha} = \frac{(1+\mathrm{e})\,|\,FQ\,|}{(1+\mathrm{e}\,\cos\alpha)^2}\left[-\,e_1\sin\alpha + e_2\left(\mathrm{e}+\cos\alpha\right)\right]$$

This derivative has the direction of the line tangent to the conic at the point *P*, its unitary vector *t* being:

$$t = \frac{-\,e_1\sin\alpha + e_2\left(\mathrm{e}+\cos\alpha\right)}{\sqrt{1+\mathrm{e}^2+2\,\mathrm{e}\,\cos\alpha}}$$

The unitary normal vector *n* is orthogonal to the tangent vector:

$$n = \frac{e_1\left(\mathrm{e}+\cos\alpha\right) + e_2\sin\alpha}{\sqrt{1+\mathrm{e}^2+2\,\mathrm{e}\,\cos\alpha}}$$

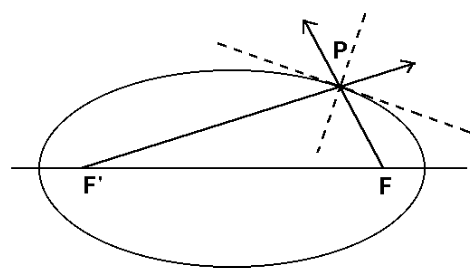
In the same way, *t* and *n* are obtained as functions of the angle $\beta$ (figure 11.9) from the vectorial equation for the focus *F'*:

$$F'P = \frac{(1-e)|FQ|}{1 - e\cos\beta}\left( e_1 \cos\beta + e_2 \sin\beta \right)$$

$$t = \frac{-e_1 \sin\beta + e_2(-e + \cos\beta)}{\sqrt{1 + e^2 - 2e\cos\beta}}$$

$$n = \frac{e_1(-e + \cos\beta) + e_2 \sin\beta}{\sqrt{1 + e^2 - 2e\cos\beta}}$$

Figure 11.10




Let us see now that the normal vector has the direction of the bisector of the angle between both focal radii (figure 11.10). In order to prove this statement, let us consider an infinitesimal displacement of a point $P$ on the conic (figure 11.11). The sum of the focal radii of any conic is constant, whence it follows that the addition of their differentials are null:

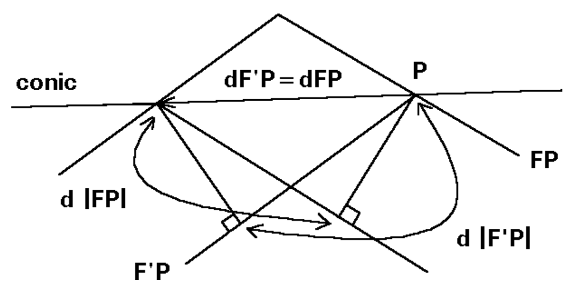
$$|FP| + |F'P| = \text{constant} \quad \Rightarrow \quad 0 = d|FP| + d|F'P|$$

The differential of the focal radius[4] is obtained by differentiating the square of the focal vector:

$$d\,FP^2 = 2\,FP \cdot d\,FP = 2\;|FP|\;d|FP|$$

$$d|FP| = -\frac{d\,FP \cdot FP}{|FP|}$$

Figure 11.11




Summing the differentials of both moduli we obtain:

$$0 = d|FP| + d|F'P| = -\frac{d\,FP \cdot FP}{|FP|} - \frac{d\,F'P \cdot F'P}{|F'P|} = -d\,FP \cdot \left( \frac{FP}{|FP|} + \frac{F'P}{|F'P|} \right)$$

since the differentials of the focal vectors are equal: $dP = d\,FP = d\,F'P$. In conclusion the differential vector (or the tangent vector) is orthogonal to the bisector of both focal vectors. The figure 11.11 clearly shows that both right triangles are opposite because they share the same hypotenuse $d\,FP$, and the legs $d|FP|$ and $d|F'P|$ have the same length. The reflection axis of both triangles, which is the shorter diagonal of the rhombus, is the bisector line of both focal vectors.

In the case of the parabola (figure 11.12), the focus $F'$ is located at the infinity and $F'P$ is parallel to the main axis of symmetry. In the case of the hyperbola (figure 11.13) we must take into account that the radius of a point on the non focal branch is negative, so

[4] Note that it differs from the modulus of the differential of the focal vector: $d|FP| \neq |d\,FP|$.

it has the sense from $P$ to the focus $F'$. The normal direction is the bisector of both oriented focal vectors.

These geometric features are the widely known optic reflection properties of the conics: parallel beams reflected by a parabola are concurrent at the focus. Also the beams emitted by a focus of an ellipse and reflected in its perimeter are concurrent in the other focus.

Figure 11.12

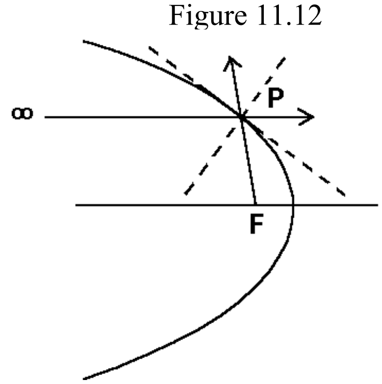
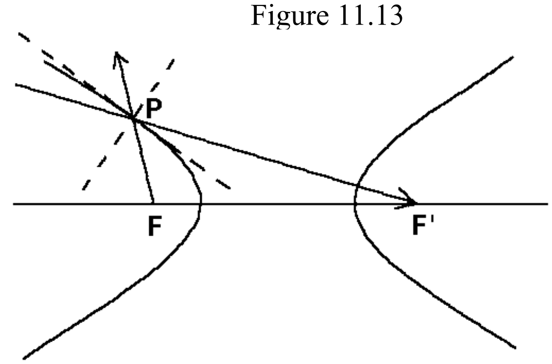



Figure 11.13




**Central equations for the ellipse and hyperbola**

We search a polar equation to describe a conic using its centre as the origin of the radius. Above we have seen that the eccentricity is the ratio of the distance between both foci and the major diameter:

$$e = \frac{|FF'|}{|QQ'|}$$

If $O$ is the centre of the conic then it is the midpoint of both foci:

$$O = \frac{F + F'}{2}$$

$$e = \frac{|OF|}{|OQ|} \qquad 1 - e = \frac{|FQ|}{|OQ|}$$

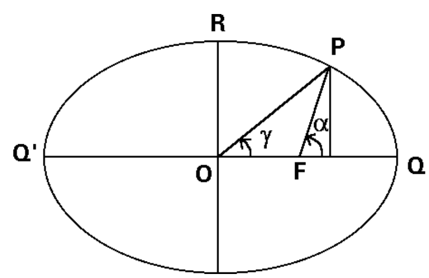



Figure 11.14

For the case of the ellipse all the quantities are positive. For the case of the hyperbola both $|OF|$ and $|OQ|$ are negative with $|FQ|$ positive. Let the angle $QOP$ be $\gamma$ (figure 11.14). Then:

$$|OP| \sin\gamma = |FP| \sin\alpha$$

$$|OP| \cos\gamma = |OF| + |FP| \cos\alpha$$

For the ellipse all the moduli are positive whereas for the hyperbola $|OP|$ and $|OF|$ are always negative (the centre is placed between both branches). For the parabola they become infinite. By substitution of the polar equation into these equalities we have:

$$|OP|\sin\gamma = |FQ|\frac{(1+\mathrm{e})\sin\alpha}{1+\mathrm{e}\cos\alpha} = |OQ|\frac{(1-\mathrm{e}^2)\sin\alpha}{1+\mathrm{e}\cos\alpha}$$

$$|OP|\cos\gamma = |OF| + \frac{|FQ|(1+\mathrm{e})\cos\alpha}{1+\mathrm{e}\cos\alpha} = |OQ|\left[\mathrm{e} + \frac{(1-\mathrm{e}^2)\cos\alpha}{1+\mathrm{e}\cos\alpha}\right] = |OQ|\frac{\mathrm{e}+\cos\alpha}{1+\mathrm{e}\cos\alpha}$$

Summing the squares of both former equalities we find:

$$OP^2\left(\frac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma\right) = OQ^2 \qquad \Rightarrow \qquad |OP| = \frac{|OQ|}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}}$$

The direction of *OP* with respect to *OQ* is given by the angle $\gamma$ and we may add it multiplying by the unitary complex with argument $\gamma$ to obtain the central vectorial equation:

$$OP = \frac{OQ\,(\cos\gamma + e_{12}\sin\gamma)}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}}$$

Let *R* be the point *P* for $\gamma=\pi/2$. Then *OR* lies on the secondary axis of symmetry of the conic, which is perpendicular to *OQ*, the main axis of symmetry. $|OR|$ is the minor half-axis. The eccentricity relates both:

$$OR^2 = (\,1-\mathrm{e}^2\,)\,OQ^2$$

$$|OR| = \sqrt{1-\mathrm{e}^2}\;|OQ|$$

$$OR = \sqrt{1-\mathrm{e}^2}\;OQ\;\;e_{12}$$

Using the minor half-axis, the equation of the ellipse becomes:

$$OP = \frac{1}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}}\left(OQ\cos\gamma + OR\frac{\sin\gamma}{\sqrt{1-\mathrm{e}^2}}\right)$$

Hence it follows:

$$\frac{OP^2}{OR^2}\sin^2\gamma + \frac{OP^2}{OQ^2}\cos^2\gamma = 1$$

Taking $OP$ and $OQ$ as the coordinates axis and $O$ the origin of coordinates, we have the canonical equation of a conic:

$$x = |OP| \cos\gamma \qquad\qquad y = |OP| \sin\gamma$$

$$\frac{x^2}{OQ^2} + \frac{y^2}{OR^2} = 1$$

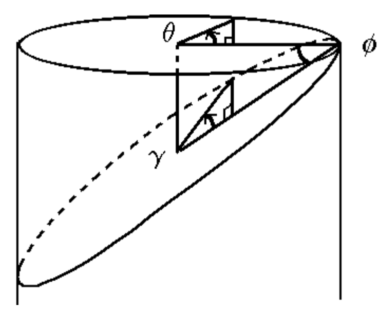
This equation of a conic is specific of this coordinate system and has a limited usefulness. If another system of coordinates is used (the general case) the equation is always of second degree but not so beautiful. For the ellipse both half-axis are positive real numbers. However for the hyperbola, the minor half-axis $|OQ|$ is imaginary and $OQ^2<0$ converting the sum of the squares in the former equation in a difference of real positive quantities.

### Diameters and Apollonius' theorem

If e<1 (ellipse) we may introduce the angle $\theta$ in the following way:

$$\cos\theta = \frac{\cos\gamma}{\sqrt{\dfrac{\sin^2\gamma}{1-e^2} + \cos^2\gamma}}$$

Figure 11.15




Then the central equation of an ellipse becomes[5]:

$$OP = OQ\cos\theta + OR\sin\theta$$

The angle $\theta$ has a direct geometric interpretation if we look at a section of a cylinder (figure 11.15):

$$\cos\theta = \frac{\cos\gamma\cos\phi}{\sqrt{\sin^2\gamma + \cos^2\gamma\cos^2\phi}}$$

From where it follows the relationship between the inclination of the cylindrical section and the eccentricity:

$$\cos\phi = \sqrt{1-e^2}$$

[5] G. Peano (*Gli elementi di calcolo geometrico* [1891] in *Opere Scelte*, vol. III, Edizioni Cremone, [Roma, 1959], p.59) gives this central equation for the ellipse as function of the angle $\theta$. He also used the parametric equations of the parabola, hyperbola, cycloid, epicycloid and the spiral of Archimedes.

Then $\theta$ is an angle between axial planes of the cylinder.

The equation of the ellipse may be written using another pair of axis turned by a cylindrical angle $\chi$ (figure 11.16):

$$OQ' = OQ\cos\chi + OR\sin\chi \qquad\qquad OR' = -OQ\sin\chi + OR\sin\chi$$

Each pair of $OQ'$ and $OR'$ obtained in this way are *conjugate central radii* and twice them are *conjugate diameters*. They are intersections of the plane of the ellipse with two axial planes of the cylinder that form a right angle. From the definition it is obvious that:

$$OQ'^{\,2} + OR'^{\,2} = OQ^2 + OR^2$$

The inverse relation between the conjugate radii are:

$$OQ = OQ'\cos\chi - OR'\sin\chi$$

$$OR = OQ'\sin\chi + OR'\cos\chi$$

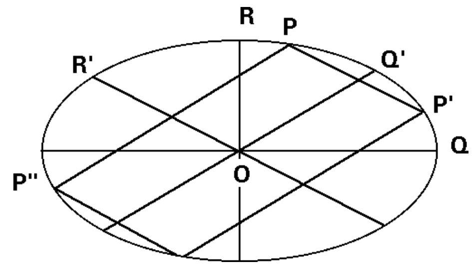



Figure 11.16

Then, the central radius of any point $P$ is:

$$OP = (\,OQ'\cos\chi - OR'\sin\chi\,)\cos\theta + (\,OQ'\sin\chi + OR'\cos\chi\,)\sin\theta$$

$$= OQ'\cos(\,\theta-\chi\,) + OR'\sin(\,\theta-\chi\,)$$

Changing the sign of the cylindrical angle we obtain another point $P'$ on the ellipse owing to the parity of trigonometric functions:

$$OP' = OQ'\cos(\theta-\chi) - OR'\sin(\theta-\chi) = OQ'\cos(\chi-\theta) + OR'\sin(\chi-\theta)$$

Then the chord $PP'$ is parallel to the radius $OR'$ (figure 11.16):

$$PP' = OP' - OP = -2\,OR'\sin(\,\theta-\chi\,)$$

Now, let us mention the Apollonius theorem: a diameter of a conic is formed by all the midpoints of the chords parallel to its conjugate diameter. To prove this statement, see that the central radius of the midpoint has the direction $OQ'$:

$$\frac{OP + OP'}{2} = OQ'\cos(\theta - \chi)$$

The properties of the conjugate diameters are also applicable to the hyperbola, but taking into account that $OR^2 = (1 - e^2)\,OQ^2 < 0$, that is, the minor half-axis $|OR|$ is an imaginary number. In this case $OR$ has the same direction than $OQ$:

$$OR = \sqrt{1-\mathrm{e}^2}\; OQ\, e_{12} = \sqrt{\mathrm{e}^2 - 1}\; OQ$$

And also the angle $\theta$ is an imaginary number, and the trigonometric functions are turned into hyperbolic functions of the real argument $\psi = \theta / e_{12}$ :

$$\cos\theta = \frac{\cos\gamma}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}} = \cosh\psi \geq 1 \qquad \mathrm{e} > 1$$

Taking into account the relations between trigonometric and hyperbolic functions:

$$\cos\frac{\psi}{e_{12}} = \cosh\psi \qquad \sin\frac{\psi}{e_{12}} = \frac{\sinh\psi}{e_{12}}$$

the relation of the hyperbolic angle $\psi$ with the Cartesian coordinates is:

$$\cosh\psi = \frac{x}{|\,OQ\,|} \qquad \sinh\psi = \frac{y}{|\,OR\,|}\, e_{12}$$

$$OP = \pm\left(OQ\cosh\psi + OR\; e_{12}\; \sinh\psi\right) = \pm\left(OQ\cosh\psi + \sqrt{e^2 - 1}\,\sinh\psi\; OQ\; e_{12}\right)$$

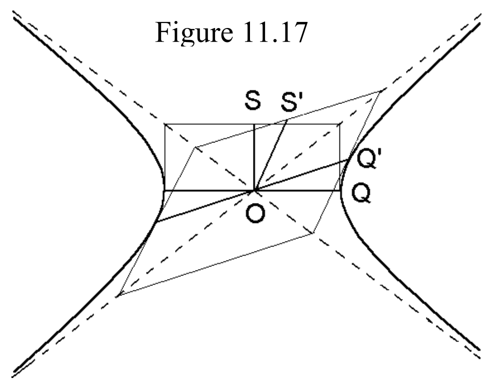
Let us define $OS$, which is taken usually as the minor half-axis of the hyperbola although this definition is not exact, as (figure 11.17):

$$OS = \sqrt{\mathrm{e}^2 - 1}\; OQ\; e_{12}$$

Then the equation of the hyperbola is:

$$OP = OQ\cosh\psi + OS\,\sinh\psi$$

Figure 11.17




When the hyperbolic angle tends to infinity we find the asymptotes:

$$\psi \to \infty \qquad OP \to \pm\frac{OQ + OS}{2}\;\exp\psi$$

$$\psi \to -\infty \qquad OP \to \pm\frac{OQ - OS}{2}\;\exp(-\psi)$$

Two radii are conjugate if they are turned through the same hyperbolic angle $\varphi$ (figure 11.17):

$$OQ' = OQ\cosh\varphi + OS\,\sinh\varphi$$

$$OS' = OQ \sinh\varphi + OS \cosh\varphi$$

Then the equality $OQ'^{\,2} - OS'^{\,2} = OQ^2 - OS^2$ holds and the central radius of any point of the hyperbola is:

$$OP = \pm\left(OQ' \cosh(\psi - \varphi) + OS' \sinh(\psi - \varphi)\right)$$

Also it is verified that the diameter is formed by the midpoints of the chords parallels to the conjugate diameter.

A complete understanding of the central equation of the hyperbola is found in the hyperbolic prism in a pseudo-Euclidean space. The equality $OQ'^{\,2} - OS'^{\,2} = OQ^2 - OS^2$ is the condition of hyperbolic prism of constant radius. Then the hyperbolas are planar section of this hyperbolic prism.

### Conic passing through five points

Five non collinear points determine a conic. Since every conic has a Cartesian second degree equation:

$$a\,x^2 + b\,y^2 + c\,x\,y + d\,x + e\,y + f = 0$$

where there are five independent parameters, the substitution of the coordinates of five points leads to a linear system with five equation, with a hard solving. A briefer way to obtain the Cartesian equation of the conic passing through these points is through the Chasles' theorem. Let us see an example:

$A = (1, 1)$ $B = (2, 3)$ $C = (1, -1)$ $D = (0, 0)$ $E = (-1, 3)$

$EA = (2, -2)$ $EB = (3, 0)$ $EC = (2, -4)$ $ED = (1, -3)$

The projective cross ratio of the four points *A, B*, *C* and *D* on the conic is:

$$r = \{E, A\,B\,C\,D\} = \frac{EA \wedge EC \;\; EB \wedge ED}{EA \wedge ED \;\; EB \wedge EC} = \frac{-4(-9)}{-4(-12)} = \frac{3}{4}$$

If the point $X = (x, y)$ then:

$XA = (1-x, 1-y)$ $XB = (2-x, 3-y)$ $XC = (1-x, -1-y)$ $XD = (-x, -y)$

According to the Chasles' theorem, the projective cross ratio is constant for any point $X$ on the conic:

$$\frac{3}{4} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} = \frac{(2x-2)(3x-2y)}{(x-y)(4x-y-5)} \quad \Rightarrow \quad 0 = 12x^2 - 3\,y^2 - x\,y - 9\,x + y$$

Another way closest to geometric algebra is as follows. Let us take *A*, *B* and *C* as a point base of the plane and express $D$ and $X$ in this base

$$D = d_A A + d_B B + d_C C \qquad \text{with} \qquad d_A + d_B + d_C = 1$$

$$X = x_A A + x_B B + x_C C \qquad \text{with} \qquad x_A + x_B + x_C = 1$$

The projective cross ratio is the quotient of the areas of the triangles $XAC$, $XBD$, $XAD$ and $XBC$:

$$r = \frac{XA \wedge XC \; XB \wedge XD}{XA \wedge XD \; XB \wedge XC} = \frac{\begin{vmatrix} x_A & x_B & x_C \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_A & x_B & x_C \\ 0 & 1 & 0 \\ d_A & d_B & d_C \end{vmatrix}}{\begin{vmatrix} x_A & x_B & x_C \\ 1 & 0 & 0 \\ d_A & d_B & d_C \end{vmatrix} \begin{vmatrix} x_A & x_B & x_C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{x_B \left( x_A\, d_C - x_C\, d_A \right)}{\left( x_B\, d_C - x_C\, d_B \right) x_A}$$

which yields the following equation:

$$0 = r\, x_C\, x_A\, d_B + (1 - r)\, x_A\, x_B\, d_C - x_B\, x_C\, d_A$$

The fifth point $E$ also lying on the conic fulfils this equation:

$$0 = r\, e_C\, e_A\, d_B + (1 - r)\, e_A\, e_B\, d_C - e_B\, e_C\, d_A$$

which results in a simplified expression for the cross ratio:

$$r = \frac{\dfrac{d_A}{e_A} - \dfrac{d_C}{e_C}}{\dfrac{d_B}{e_B} - \dfrac{d_C}{e_C}}$$

The substitution of the cross ratio $r$ in the equation of the conic gives:

$$d_A\, e_A\, x_B\, x_C \left( d_B\, e_C - d_C\, e_B \right) + d_B\, e_B\, x_C\, x_A \left( d_C\, e_A - d_A\, e_C \right) + d_C\, e_C\, x_A\, x_B \left( d_A\, e_B - d_B\, e_A \right) =$$

$$\begin{vmatrix} d_A\, e_A & d_B\, e_B & d_C\, e_C \\ d_A\, x_A & d_B\, x_B & d_C\, x_C \\ e_A\, x_A & e_B\, x_B & e_C\, x_C \end{vmatrix} = 0$$

**Conic equations in barycentric coordinates and tangential conic**

The Cartesian equation of a conic:

$$a\, x^2 + b\, y^2 + c\, x\, y + d\, x + e\, y + f = 0$$

is written in barycentric coordinates as a bilinear mapping:

$$\begin{pmatrix}1-x-y & x & y\end{pmatrix}\begin{pmatrix} f & \frac{d}{2}+f & \frac{e}{2}+f \\ \frac{d}{2}+f & a+d+f & \frac{c+d+e}{2}+f \\ \frac{e}{2}+f & \frac{c+d+e}{2}+f & b+e+f \end{pmatrix}\begin{pmatrix}1-x-y \\ x \\ y\end{pmatrix}=0$$

Let us calculate now the dual conic of a given conic. This is defined as the locus of the points which are dual of the lines tangent to the conic. Then the conic is the envelope of the tangents. Let us differentiate the Cartesian equation:

$$\delta x\,(2\,a\,x+c\,y+d)+\delta y\,(2\,b\,y+c\,x+e)=0$$

where $\delta$ indicates the ordinary differential in order to avoid confusion with the coefficient *d*. The equation of the line touching the conic at $(x_0, y_0)$ is:

$$(2\,a\,x_0+c\,y_0+d)(x-x_0)+(2\,b\,y_0+c\,x_0+e)(y-y_0)=0$$

$$(2\,a\,x_0+c\,y_0+d)\,x+(2\,b\,y_0+c\,x_0+e)\,y-2\,a\,x_0^2-2\,b\,y_0^2-2\,c\,x_0\,y_0-d\,x_0-e\,y_0=0$$

$$(2\,a\,x_0+c\,y_0+d)\,x+(2\,b\,y_0+c\,x_0+e)\,y+d\,x_0+e\,y_0+2f=0$$

$$\begin{aligned}&(d\,x_0+e\,y_0+2f)(1-x-y)+((2\,a+d)\,x_0+(c+e)\,y_0+d+2\,f)\,x+\\ &\quad+((2\,b+e)\,y_0+(c+d)\,x_0+e+2f)\,y=0\end{aligned}$$

If we denote by *t*, *u* and *v* the dual coordinates, the dual homogeneous coordinates *t'*, *u'* and *v'* are linear functions of the barycentric point coordinates:

$$\begin{bmatrix}t' \\ u' \\ v'\end{bmatrix}=\begin{pmatrix} 2\,f & d+2\,f & e+2\,f \\ d+2\,f & 2\,a+2\,d+2\,f & c+d+e+2\,f \\ e+2\,f & c+d+e+2\,f & 2\,b+2\,e+2\,f \end{pmatrix}\begin{pmatrix}1-x_0-y_0 \\ x_0 \\ y_0\end{pmatrix}$$

Now we see that this is twice the matrix of the conic equation (the matrix of a conic is defined except by a common factor). Let **M** be the matrix of the conic, **X** the matrix of the point coordinates, and **U** the matrix of the dual (homogeneous or normalised) coordinates.

$$\mathbf{U}=\mathbf{M}\,\mathbf{X}$$

$$\mathbf{M}^{-1}\,\mathbf{U}=\mathbf{X}$$

$$\mathbf{X}^{\mathrm{T}}=\mathbf{U}^{\mathrm{T}}\,\mathbf{M}^{-1}$$

because like **M**, the inverse matrix $\mathbf{M}^{-1}$ is also symmetric and does not change under transposition. The substitution in the equation of the conic $\mathbf{X}^{T}\,\mathbf{M}\,\mathbf{X} = 0$ gives:

$$\mathbf{U}^{T}\,\mathbf{M}^{-1}\,\mathbf{U} = 0$$

That is, <u>the matrix of the dual conic equation is the inverse of the matrix of the point conic</u>. Every tangent line of the conic is mapped onto a point of the dual conic and vice versa. Let us consider for instance the ellipse:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \qquad \Leftrightarrow \qquad \begin{pmatrix} 1-x-y & x & y \end{pmatrix} \begin{pmatrix} 16 & 16 & 16 \\ 16 & 15 & 16 \\ 16 & 16 & 12 \end{pmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

The inverse matrix is:

$$\begin{pmatrix} 16 & 16 & 16 \\ 16 & 15 & 16 \\ 16 & 16 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} -19/16 & 1 & 1/4 \\ 1 & -1 & 0 \\ 1/4 & 0 & -1/4 \end{pmatrix}$$

So the equation of the dual conic is:

$$\begin{bmatrix} t & u & v \end{bmatrix} \begin{pmatrix} -19 & 16 & 4 \\ 16 & -16 & 0 \\ 4 & 0 & -4 \end{pmatrix} \begin{bmatrix} t \\ u \\ v \end{bmatrix} = 0$$

where we can substitute indistinctly homogeneous or normalised coordinates. If we take $t = 1 - u - v$ then we obtain the Cartesian equation:

$$-67\,u^2 - 31\,v^2 - 78\,u\,v + 70\,u + 46\,v - 19 = 0$$

A concrete case is the tangent line at (0, 2) with equation $y = 2$. The dual coordinates of this line are obtained as follows:

$$y - 2 \equiv t'\,(1 - x - y) + u'\,x + v'\,y \qquad \Rightarrow \qquad \begin{bmatrix} t' & u' & v' \end{bmatrix} = \begin{bmatrix} -2, -2, -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5}, \frac{1}{5} \end{bmatrix}$$

Let us check that this dual point lies on the dual conic:

$$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{pmatrix} -19 & 16 & 4 \\ 16 & -16 & 0 \\ 4 & 0 & -4 \end{pmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 0$$

### Polarities

A *correlation* is defined as the geometric transformation which maps collinear points onto a pencil of lines, in other words, a linear mapping of points to lines (dual points), which may be represented with a matrix **M**:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \mathbf{M} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix}$$

where $a'$, $b'$ and $c'$ are homogeneous dual coordinates.

A *polarity* is a correlation whose square is the identity. This means that by applying it twice, a point is mapped onto itself and a line is mapped onto itself. Let us consider the line $[a, b, c]$:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} a & b & c \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = 0$$

On the other hand:

$$\begin{bmatrix} a' & b' & c' \end{bmatrix} \begin{pmatrix} 1-x'-y' \\ x' \\ y' \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} a' & b' & c' \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

since $[\, a'' \;\; b'' \;\; c'' \,] = k\, [\, a \;\; b \;\; c \,]$ where $k$ is a homogeneous constant. Whence it follows that the matrix **M** is symmetric for a polarity.

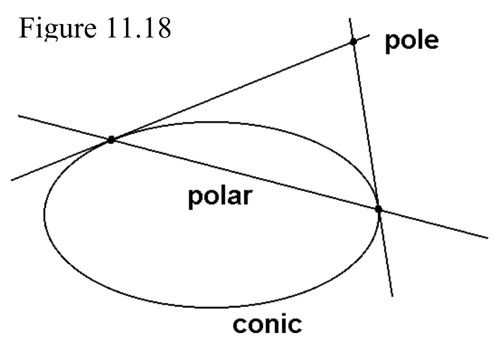
Under a polarity a point is mapped to its *polar line*. This point is also called the *pole* of the line. The polarity transforms the pole into the polar and the polar into the pole. In general the polar does not include the pole, except by a certain subset of points on the plane. The set of points belonging to their own polar line (*self-conjugate points)* fulfil the equation:

$$\begin{pmatrix} 1-x-y & x & y \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

This is just the equation of a point conic. The set of lines passing through their own poles (*self-conjugate lines*) fulfil the equation:

$$\begin{bmatrix} a & b & c \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Figure 11.18




which is just the equation of its dual (*tangential*) conic. Then, a polarity has an associated point and tangential conics, and every conic defines a polarity. However depending on the eigenvalues of the matrix, there are polarities that do not have any associated conic.

The polar of a point outside its associated conic is the line passing through the points of contact of the tangents drawn from the point (figure 11.18). The reason is the following: the polarity maps the tangent lines of the associated conic to the tangency points, so its intersection (the pole) must be transformed into the line passing through the tangency points.

### Reduction of the conic matrix to a diagonal form

Since the matrix of a polarity is symmetric, we can reduce it to a diagonal form. Let **D** be the diagonal matrix and **B** the exchange base matrix. Then:

$$\mathbf{M} = \mathbf{B}^{-1}\,\mathbf{D}\,\mathbf{B}$$

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

There exist three eigenvectors of the conic matrix. These eigenvectors are also characteristic vectors of the dual conic matrix with inverse eigenvalues:

$$\mathbf{M}^{-1} = \mathbf{B}^{-1}\,\mathbf{D}^{-1}\,\mathbf{B}$$

$$\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix} \qquad\qquad d_i\,\mathbf{V}_i = \mathbf{M}\,\mathbf{V}_i \quad \Leftrightarrow \quad \mathbf{M}^{-1}\,\mathbf{V}_i = \frac{1}{d_i}\,\mathbf{V}_i$$

The eigenvectors are three points (equal for both punctual and dual planes) that I will call the *eigenpoints* of the polarity. Let **X** be a point on the conic given in Cartesian coordinates. Then:

$$\mathbf{X}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{X} = \mathbf{X}^{\mathrm{T}}\,\mathbf{B}^{-1}\,\mathbf{D}\,\mathbf{B}\,\mathbf{X} = \mathbf{V}^{\mathrm{T}}\,\mathbf{D}\,\mathbf{V}$$

where $\mathbf{V} = \mathbf{B}\,\mathbf{X}$ are the coordinates of the point in the base of eigenpoints. If we name with $t$, $u$, $v$ the eigencoordinates, then the equation of the conic is simply:

$$d_1\,t^2 + d_2\,u^2 + d_3\,v^2 = 0$$

The eigenpoints of a polarity do not never belong to the associated conic since they have the eigencoordinates (1,0,0), (0,1,0) and (0,0,1) and $d_i \neq 0$. On the other hand, the conic only exist if the eigenvalues have different signs (one positive and two negative or one negative and two positive). This fact classifies the polarities in two classes: those having an associated conic, and those not having an associated conic. In fact, if we state a level, a certain value of the matrix product, there can be a conic or not. That is, there is a family of conics for different levels and there is a threshold value, from where the conic does not exist.

### Using a base of points on the conic

If instead of Cartesian coordinates we use the coordinates of a base of three points $A$, $B$ and $C$ lying on the conic, the coefficients on the matrix diagonal vanish. Then the equation of the conic is:

$$\frac{1}{2}\begin{pmatrix} x_A & x_B & x_C \end{pmatrix}\begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{12} & 0 & c_{23} \\ c_{13} & c_{23} & 0 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = c_{23}\, x_B\, x_C + c_{31}\, x_C\, x_A + c_{12}\, x_A\, x_B = 0$$

With the same base, the equation of the conic passing through $D$ and whose projective cross ratio $\{X, A\ B\ C\ D\}$ is $r$ is:

$$0 = r\, x_C\, x_A\, d_B + (1-r)\, x_A\, x_B\, d_C - x_B\, x_C\, d_A$$

From where we have:

$$\begin{pmatrix} x_A & x_B & x_C \end{pmatrix}\begin{pmatrix} 0 & (1-r)\, d_C & r\, d_B \\ (1-r)\, d_C & 0 & -d_A \\ r\, d_B & -d_A & 0 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = 0$$

This conic matrix is not constant, and has the determinant:

$$\det = -2\, r\,(1-r)\, d_A\, d_B\, d_C$$

so the matrix with constant coefficients is:

$$\mathbf{C} = \frac{1}{\sqrt[3]{-r\,(1-r)\, d_A\, d_B\, d_C}}\begin{pmatrix} 0 & (1-r)\, d_C & r\, d_B \\ (1-r)\, d_C & 0 & -d_A \\ r\, d_B & -d_A & 0 \end{pmatrix}$$

### Exercises

11.1 Prove that the projective cross ratio of any four points on a conic is equal to the ratio of the sines of the cylindrical half-angles. That is, if:

$OA = OQ \cos\alpha + OR \sin\alpha$ $\qquad$ $OB = OQ \cos\beta + OR \sin\beta$, etc.

Then $\{X, A\,B\,C\,D\} = \dfrac{\sin\dfrac{\gamma-\alpha}{2}\sin\dfrac{\delta-\beta}{2}}{\sin\dfrac{\delta-\alpha}{2}\sin\dfrac{\gamma-\beta}{2}}$

11.2 Prove that an affinity transforms a circle into an ellipse, and using this fact prove the Newton's theorem: Given two directions, the ratio of the product of distances from any point on the plane to both intersections with an ellipse along the first direction and the same product of distances along the other direction is independent of the location of this point on the plane.

11.3 Prove that a line touching an ellipse is parallel to the conjugate diameter of the diameter whose end-point is the tangency point. Prove also that the parallelogram circumscribed around an ellipse has constant area (Apollonius' theorem).

11.4 Prove the former theorem applied to a hyperbola: the parallelogram formed by two conjugate diameters has constant area.

11.5 Given a pole, trace a line passing through the pole and cutting the conic. Prove that the pole, and the intersections of this line with the conic and with the polar form a harmonic range.

11.6 Calculate the equation of the point and tangential conic passing through the points (1, 1), (2, 3), (1, 4), (0, 2) and (2, 5).

11.7 The Pascal's theorem: Let *A*, *B*, *C*, *D*, *E* and *F* be six distinct points on a proper conic, Let *P* be the intersection of the line *AE* with the line *BF*, *Q* the intersection of the line *AD* with *CF*, and *R* the intersection of the line *BD* with *CE*. Show that *P*, *Q* and *R* are collinear.

11.8 The Brianchon's theorem. Prove that the diagonals joining opposite vertices of a hexagon circumscribed around a proper conic are concurrent. Hint: take the dual plane.