

THIRD PART: PSEUDO-EUCLIDEAN GEOMETRY

This part is devoted to the hyperbolic plane, where vectors and numbers have a pseudo Euclidean modulus, that is, a modulus of a Minkowski space. The bidimensional geometric algebra already includes the Euclidean and pseudo Euclidean planes. In fact, the geometric algebra does not make any special distinction between both kinds of planes. On the other hand, the plane geometric algebra can be represented by real 2×2 matrices, which helps us to define some concepts with more precision.

12. MATRIX REPRESENTATION AND HYPERBOLIC NUMBERS

All the associative algebras with neutral elements for the addition and product can be represented with matrices. The representations can have different dimensions, but the most interesting is the minimal representation which is an isomorphism.

Rotations and the representation of complex numbers

We have seen that a rotation of a vector v over an angle α is written in geometric algebra as:

$$v' = v (\cos \alpha + e_{12} \sin \alpha)$$

Separating the components:

$$v_1' = v_1 \cos \alpha - v_2 \sin \alpha$$

$$v_2' = v_1 \sin \alpha + v_2 \cos \alpha$$

and writing them in matrix form, we have:

$$\begin{pmatrix} v_1' & v_2' \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

Identifying this equation with the first one:

$$\cos \alpha + e_{12} \sin \alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

we obtain the matrix representation for the complex numbers:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad z = a + b e_{12} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Now we wish to obtain also the matrix representation for vectors. Note that the matrix form of the rotation of a vector gives us the first row of the vector representation, which may be completed with a suitable second row:

$$\begin{pmatrix} v_1' & v_2' \\ v_2' & -v_1' \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

from where we arrive to the matrix representation for vectors:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad v = v_1 e_1 + v_2 e_2 = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$$

These four matrices are a basis of the real matrix space $\mathbf{M}_{2 \times 2}(\mathbf{R})$ and therefore the geometric algebra of the vectorial plane V_2 is isomorphic to this matrix space:

$$Cl(V_2) \equiv Cl_{2,0} = \mathbf{M}_{2 \times 2}(\mathbf{R})$$

The moduli of a complex number and a vector are related with the determinant of the matrix:

$$|z|^2 = a^2 + b^2 = \det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad |v|^2 = v_1^2 + v_2^2 = -\det \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$$

Let us think about a heterogeneous element being a sum of a complex and a vector. Its geometric meaning is unknown for us but we can write it with a matrix:

$$z + v = a + b e_{12} + v_1 e_1 + v_2 e_2 = \begin{pmatrix} a + v_1 & b + v_2 \\ -b + v_2 & a - v_1 \end{pmatrix}$$

The determinant of this matrix

$$\det(z + v) = a^2 + b^2 - v_1^2 - v_2^2$$

reminds us the Pythagorean theorem in a pseudo Euclidean plane.

The subalgebra of the hyperbolic numbers

In the matrix representation of the geometric algebra, we can see that the diagonal matrices are a subalgebra:

$$a + b e_1 = \begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix}$$

The product of elements of this kind is commutative (and also associative). Because of this, we may talk about «numbers»:

$$(a + b e_1)(c + d e_1) = ac + bd + (ad + bc) e_1$$

Using the Hamilton's notation of pairs of real numbers, we write:

$$(a, b)(c, d) = (ac + bd, ad + bc)$$

From the determinant we may define their pseudo Euclidean *modulus* $|z|$:

$$\det(a + b e_1) = a^2 - b^2 = |a + b e_1|^2$$

The numbers having constant modulus lie on a hyperbola. Because of this, they are called *hyperbolic numbers*. Due to the fact that e_1 cannot be a privileged direction on the plane, any other set of elements having the form:

$$a + k u$$

with a and k real, and u being a fixed unitary vector, is also a set of hyperbolic numbers.

By defining the *conjugate* of a hyperbolic number as that number with the opposite vector component:

$$(a + b e_1)^* = a - b e_1$$

we see that the square of the modulus of a hyperbolic number is equal to the product of this number by its conjugate:

$$|z|^2 = z z^*$$

Hence the inverse of a hyperbolic number follows:

$$z^{-1} = \frac{z^*}{|z|^2}$$

Since the modulus is the square root of the determinant, and the determinant of a matrix product is equal to the product of determinants, it follows that the modulus of a product of hyperbolic numbers is equal to the product of moduli:

$$|z t| = |z| |t|$$

The elements $(1 + e_1)/2$ and $(1 - e_1)/2$, which are idempotents, and their multiples have a null modulus and no inverse. They form two ideals, that is, the product of any hyperbolic number by a multiple of an idempotent (or any other multiple) yields also a multiple of the idempotent.

Hyperbolic trigonometry

Let us consider the locus of the points (hyperbolic numbers) located at a constant distance r from the origin (hyperbolic numbers with constant determinant equal to r^2), which lie on the hyperbola $x^2 - y^2 = r^2$ (figure 12.1). I shall call r the *hyperbolic radius*

following the analogy with the circle. The extreme of the radius is a point on the hyperbola with coordinates (x, y) . The arc between the positive X half axis and this point (x, y) has an oriented length s . On the other hand, the radius, the hyperbola and the X -axis delimit a sector with an oriented area A . The hyperbolic angle (or argument) ψ is defined as the quotient of the arc length divided by the radius¹:

$$\psi = \frac{s}{r}$$

It follows from this definition that the oriented area A is²:

$$A = \frac{\psi r^2}{2} \Rightarrow \psi = \frac{2A}{r^2}$$

The hyperbolic sine, cosine and tangent are defined as the following quotients:

$$\begin{aligned} \sinh \psi &= \frac{y}{r} \\ \cosh \psi &= \frac{x}{r} & \text{tgh } \psi &= \frac{y}{x} \end{aligned}$$

These definitions yield the three fundamental identities of the hyperbolic trigonometry:

$$\text{tgh } \psi \equiv \frac{\sinh \psi}{\cosh \psi} \qquad \cosh^2 \psi - \sinh^2 \psi \equiv 1 \qquad 1 - \text{tgh}^2 \psi \equiv \frac{1}{\cosh^2 \psi}$$

Now we search an explicit expression of the hyperbolic functions in terms of elemental functions such as polynomials, exponential, logarithm, etc. The differential of the arc length (being real) is related with the differentials of the coordinates by the pseudo Pythagorean theorem:

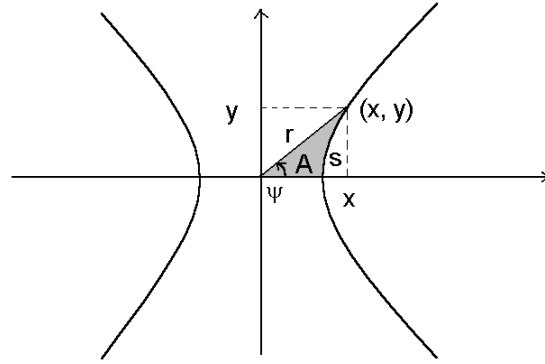
$$ds^2 = dy^2 - dx^2 \quad \Rightarrow \quad 1 = \left(\frac{d(\sinh \psi)}{d\psi} \right)^2 - \left(\frac{d(\cosh \psi)}{d\psi} \right)^2$$

That is, we have the following system with one differential equation:

¹ Note that $x^2 - y^2 = r^2 > 0$ while $s^2 < 0$. However I overcome this trouble taking in these definitions r and s and also the area A as real numbers but being oriented, that is, with sign.

² The formula of the sector area is obtained by an analogous argument from Archimedes: the total area is the addition of areas of the infinitesimal triangles with altitude equal to r and base equal to ds . r being constant, the area of the hyperbola sector is $A = r s / 2$. Obviously the radius is orthogonal to each infinitesimal piece of arc of the hyperbola. This question and the concept and calculus of areas are studied with more detail in the following chapter.

Figure 12.1



$$\begin{cases} 1 = \cosh^2 \psi - \sinh^2 \psi \\ 1 = (\sinh' \psi)^2 - (\cosh' \psi)^2 \end{cases}$$

whose solution, according to the initial conditions given by the geometric definition ($\sinh 0 = 0$ and $\cosh 0 = 1$), is:

$$\sinh \psi = \frac{\exp \psi - \exp(-\psi)}{2} \qquad \cosh \psi = \frac{\exp \psi + \exp(-\psi)}{2}$$

Hyperbolic exponential and logarithm

Within the hyperbolic numbers one can define and study functions and develop a *hyperbolic analysis*, with the aid of matrix functions. In the matrix algebra, the exponential function of a matrix A is defined as:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

The matrices which represent the hyperbolic numbers (e_1 direction) are diagonal. Then their exponential matrix has the exponential of each element in the diagonal:

$$\exp(x + y e_1) = \exp \begin{pmatrix} x + y & 0 \\ 0 & x - y \end{pmatrix} = \begin{pmatrix} \exp(x + y) & 0 \\ 0 & \exp(x - y) \end{pmatrix}$$

By extracting the common factor and introducing the hyperbolic functions we arrive at:

$$\exp(x + y e_1) = \exp(x) \begin{pmatrix} \cosh y + \sinh y & 0 \\ 0 & \cosh y - \sinh y \end{pmatrix}$$

which is the analogous of the Euler's identity:

$$\exp(x + y e_1) = \exp(x) (\cosh y + e_1 \sinh y)$$

From this exponential identity, we can find the logarithm function:

$$\log(x + y e_1) = \frac{1}{2} \log(x^2 - y^2) + e_1 \arg \operatorname{tgh} \left(\frac{y}{x} \right) = \frac{1}{2} \log(x^2 - y^2) + \frac{e_1}{2} \log \left(\frac{x + y}{x - y} \right)$$

for $-x < y < x$. This condition is the set of hyperbolic numbers with positive determinant, which is called a *sector*³. The two ideals generated by the idempotents $(1 + e_1)/\sqrt{2}$ and $(1 - e_1)/\sqrt{2}$ separate two sectors of hyperbolic numbers, one with

³ According to the relativity theory, the region of the space-time accessible to our knowledge must fulfil the inequality $c^2 t^2 - x^2 \geq 0$, being x the space coordinate, t the time and c the light celerity.

positive determinant and real modulus and another with negative determinant and imaginary modulus.

The characteristic property of the exponential function is:

$$\exp(z + t) = \exp(z) \exp(t)$$

z, t being hyperbolic numbers. Taking exponentials with unity determinant:

$$\exp((\psi + \chi)e_1) \equiv \exp(\psi e_1) \exp(\chi e_1)$$

and splitting the real and vectorial parts, we obtain the addition identities:

$$\cosh(\psi + \chi) \equiv \cosh \psi \cosh \chi + \sinh \psi \sinh \chi$$

$$\sinh(\psi + \chi) \equiv \sinh \psi \cosh \chi + \cosh \psi \sinh \chi$$

Also through the equality:

$$\exp(n \psi e_1) \equiv (\exp(\psi e_1))^n$$

the analogous of Moivre's identity is found:

$$\cosh(n\psi) + e_1 \sinh(n\psi) \equiv (\cosh \psi + e_1 \sinh \psi)^n$$

For example, for $n = 3$ it becomes:

$$\cosh 3\psi = \cosh^3 \psi + 3 \cosh \psi \sinh^2 \psi$$

$$\sinh 3\psi = 3 \cosh^2 \psi \sinh \psi + \sinh^3 \psi$$

Polar form, powers and roots of hyperbolic numbers

The exponential allows us to write any hyperbolic number in polar form. For example:

$$z = 13 + 5 e_1 \quad |z| = \sqrt{13^2 - 5^2} = 12 \quad \arg z = \arg \operatorname{tgh} \frac{13}{5} = \log \frac{3}{2}$$

$$z = 12 \exp\left(e_1 \log \frac{3}{2}\right) = 12 \left(\cosh\left(\log \frac{3}{2}\right) + e_1 \sinh\left(\log \frac{3}{2}\right) \right) = 12_{\log \frac{3}{2}}$$

The product of hyperbolic numbers, like that of complex numbers, is found by the multiplication of the moduli and the addition of the arguments, while the division is obtained as the quotient of moduli and difference of arguments. The division is not possible when the denominator is a multiple of an idempotent (modulus zero).

The power of a hyperbolic number has modulus equal to the power of the modulus, and argument the addition of arguments. Here the periodicity is absent, and distinct arguments always correspond to distinct numbers. The roots must be viewed with more detail. For example, there are four square roots of $13 + 12 e_1$:

$$13 + 12 e_1 = (3 + 2 e_1)^2 = (2 + 3 e_1)^2 = (-3 - 2 e_1)^2 = (-2 - 3 e_1)^2$$

that is, the equation of second degree $z^2 - 13 - 12 e_1 = 0$ has four solutions. Recently G. Casanova⁴ has proved that n^2 is the maximum number of hyperbolic solutions (including real values) of an algebraic equation of n^{th} degree. In the case of the equation of second degree we can use the classical formula whenever we know to solve the square roots. For example, let us consider the equation:

$$z^2 - 5z + 5 + e_1 = 0$$

whose solutions, according to the second degree formula, are:

$$z = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot (5 + e_1)}}{2 \cdot 1} = \frac{5 \pm \sqrt{5 - 4 e_1}}{2}$$

Now, we must calculate all the square roots of the number $5 - 4 e_1$. In order to find them we obtain its modulus and argument:

$$|5 - 4 e_1| = \sqrt{5^2 - (-4)^2} = 3 \quad \arg(5 - 4 e_1) = \arg \operatorname{tgh} \frac{-4}{5} = -\log 3$$

The square root has half argument and the square root of the modulus:

$$\sqrt{5 - 4 e_1} = \sqrt{3} \cdot \frac{\log 3}{2} = \sqrt{3} [\cosh(-\log \sqrt{3}) + e_1 \sinh(-\log \sqrt{3})] = 2 - e_1$$

There are four square roots of this number:

$$2 - e_1, -2 + e_1, 1 - 2 e_1, -1 + 2 e_1$$

and so four solutions of the initial equation:

$$\begin{aligned} z_1 &= \frac{5 + 2 - e_1}{2} = \frac{7 - e_1}{2} & z_2 &= \frac{5 - 2 + e_1}{2} = \frac{3 + e_1}{2} \\ z_3 &= \frac{5 + 1 - 2 e_1}{2} = 3 - e_1 & z_4 &= \frac{5 - 1 + 2 e_1}{2} = 2 + e_1 \end{aligned}$$

Let us study the second degree equation with real coefficients using the matrix representation:

⁴ Gaston Casanova, *Advances in Applied Clifford Algebras* **9**, 2 (1999) 215-219.

$$a z^2 + b z + c = 0 \quad a, b \text{ and } c \text{ real} \quad \text{and } z = x + e_1 y$$

Dividing the second degree equation by a we obtain the equivalent equation:

$$z^2 + \frac{b}{a} z + \frac{c}{a} = 0$$

which is fulfilled by the number z , but also by its matrix representation. So, according to the Hamilton-Cayley theorem, it is the characteristic polynomial of the matrix of z , whose eigenvalues are the elements on the diagonal $x + y$, $x - y$ (whenever they are distinct, that is, z is not real). Since b is the opposite of the sum of eigenvalues, and c is their product, we have:

$$z^2 - 2x z + x^2 - y^2 = 0$$

which gives the equalities:

$$x = -\frac{b}{2a} \quad \frac{c}{a} = x^2 - y^2 \quad \Rightarrow \quad y^2 = \frac{b^2 - 4ac}{4a^2}$$

and the solutions:

$$z = \frac{-b \pm e_1 \sqrt{b^2 - 4ac}}{2a}$$

The two real solutions must be added to these values, obtaining the four expected. Anyway, the equation has only solutions if the discriminant is positive. Let us see an example:

$$z^2 + 3z + 2 = 0$$

$$z_1 = \frac{-3+1}{2} = -1 \quad z_2 = \frac{-3-1}{2} = -2 \quad z_3 = \frac{-3+e_1}{2} \quad z_4 = \frac{-3-e_1}{2}$$

On the other hand, we may calculate for instance the cubic root of $14-13e_1$. The modulus and argument of this number are:

$$|14 - 13e_1| = \sqrt{14^2 - 13^2} = \sqrt{27} \quad \arg(14 - 13e_1) = \arg \operatorname{tgh}\left(-\frac{13}{14}\right) = -\log \sqrt{27}$$

From where it follows that the unique cubic root is:

$$\sqrt[3]{14 - 13e_1} = \sqrt{3} \cdot \frac{\log \sqrt{27}}{3} = \sqrt{3} \cdot \log \sqrt{3} = 2 - e_1$$

In order to study with more generality the number of roots, let us consider again the matrix representation of a hyperbolic number:

$$a + b e_1 = \begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix}$$

Now, it is obvious that there is only a unique root with odd index, which always exists for every hyperbolic number:

$$\sqrt[n]{a + b e_1} = \begin{pmatrix} \sqrt[n]{a + b} & 0 \\ 0 & \sqrt[n]{a - b} \end{pmatrix} \quad n \text{ odd}$$

On the other hand, if $a + b > 0$ and $a - b > 0$ (first half-sector) there are four roots with even index, one on each half-sector (I follow the anticlockwise order as usually):

n even

$$\begin{pmatrix} \sqrt[n]{a + b} & 0 \\ 0 & \sqrt[n]{a - b} \end{pmatrix} \in 1^{\text{st}} \text{ half-sector} \quad \begin{pmatrix} \sqrt[n]{a + b} & 0 \\ 0 & -\sqrt[n]{a - b} \end{pmatrix} \in 2^{\text{nd}} \text{ half-sector}$$

$$\begin{pmatrix} -\sqrt[n]{a + b} & 0 \\ 0 & -\sqrt[n]{a - b} \end{pmatrix} \in 3^{\text{rd}} \text{ half-sector} \quad \begin{pmatrix} -\sqrt[n]{a + b} & 0 \\ 0 & \sqrt[n]{a - b} \end{pmatrix} \in 4^{\text{th}} \text{ half-sector}$$

If the number belongs to a half-sector different from the first, some of the elements on the diagonal is negative and there is not any even root. This shows a panorama of the hyperbolic algebra far from that of the complex numbers.

Hyperbolic analytic functions

Which conditions must fulfil a hyperbolic function $f(z)$ of a hyperbolic variable z to be analytic? We wish that the derivative be well defined:

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

that is, this limit must be independent of the direction of Δz . If $f(z) = a + b e_1$ and the variable $z = x + y e_1$, then the derivative calculated in the direction $\Delta z = \Delta x$ is:

$$f'(z) = \frac{\partial a}{\partial x} + e_1 \frac{\partial b}{\partial x}$$

while the derivative calculated in the direction $\Delta z = e_1 \Delta y$ becomes:

$$f'(z) = e_1 \frac{\partial a}{\partial y} + \frac{\partial b}{\partial y}$$

Both expressions must be equal, which results in the *conditions of hyperbolic analyticity*:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \quad \text{and} \quad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$$

Note that the exponential and logarithm fulfil these conditions and therefore they are hyperbolic analytic functions. More exactly, the exponential is analytic in all the plane while the logarithm is analytic for the sector of hyperbolic numbers with positive determinant.

By derivation of both identities one finds that the analytic functions satisfy the hyperbolic partial differential equation (also called wave equation):

$$\frac{\partial^2 a}{\partial x^2} - \frac{\partial^2 a}{\partial y^2} = \frac{\partial^2 b}{\partial x^2} - \frac{\partial^2 b}{\partial y^2} = 0$$

Now, we must state the main integral theorem for hyperbolic analytic functions: if a hyperbolic function is analytic in a certain domain on the hyperbolic plane, then its integral following a closed way C within this domain is zero. If the hyperbolic function is $f(z) = a + b e_1$ then the integral is:

$$\oint_C f(z) dz = \oint_C (a + b e_1) (dx + dy e_1) = \oint_C (a dx + b dy) + e_1 \oint_C (a dy + b dx)$$

Since C is a closed path, we may apply the Green theorem to write:

$$= \iint_D \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy + e_1 \iint_D \left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) dx \wedge dy = 0$$

where D is the region bounded by the closed way C . Since $f(z)$ fulfils the analyticity conditions everywhere within D , the integral vanishes.

From here other theorems follow like for the complex analysis, e. g.: if $f(z)$ is a hyperbolic analytic function in a domain D simply connected and z_1 and z_2 are two points on D then the definite integral:

$$\int_{z_2}^{z_1} f(z) dz$$

between these points has a unique value independently of the integration path.

Let us see an example. Consider the function $f(z) = 1 / (z - 1)$. The function is only defined if the inverse of $z - 1$ exists, which implies $|z - 1| \neq 0$. Of course, this function is not analytic at $z = 1$, but neither for the points:

$$|z - 1|^2 = 0 \Leftrightarrow (x - 1)^2 - y^2 = 0 \Leftrightarrow (x + y - 1)(x - y - 1) = 0$$

The lines $x + y = 1$ and $x - y = 1$ break the analyticity domain into two sectors. Let us calculate the integral:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1}$$

through two different trajectories within a sector. The first one is a straight path given by the parametric equation $z = 5 + t e_1$ (figure 12.2):

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \int_{-3}^3 \frac{dt}{4+t e_1} e_1 = \int_{-3}^3 \frac{(4-t e_1) dt}{16-t^2} e_1$$

Ought to symmetry the integral of the odd function is zero:

$$= \int_{-3}^3 \frac{4 dt}{16-t^2} e_1 = \left[\frac{e_1}{2} \log \left| \frac{4+t}{4-t} \right| \right]_{-3}^3 = e_1 \log 7$$

The second path (figure 12.2) is the hyperbola going from the point $(5, -3)$ to $(5, 3)$:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \int_{(5,-3)}^{(5,3)} \frac{(z^* - 1) dz}{(z-1)(z^* - 1)} = \int_{(5,-3)}^{(5,3)} \frac{(z^* - 1) dz}{\det(z) - 2 \operatorname{Re}(z) + 1}$$

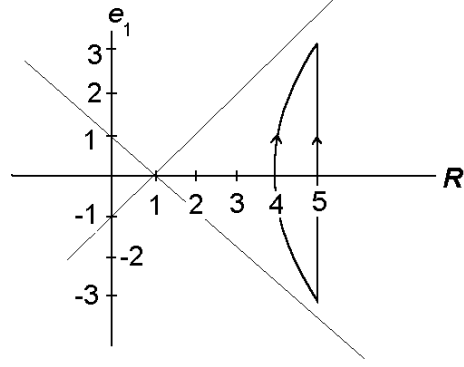


Figure 12.2

Introducing the parametric equation of this path, $z = 4 (\cosh t + e_1 \sinh t)$ we have:

$$= \int_{-\log 2}^{\log 2} \frac{(4 \cosh t - 4 e_1 \sinh t - 1)(4 \sinh t + 4 e_1 \sinh t)}{16 - 8 \cosh t + 1} dt = 4 \int_{-\log 2}^{\log 2} \frac{e_1 (4 - \cosh t) - 4 \sinh t}{17 - 8 \cosh t} dt$$

Due to symmetry, the integral of the hyperbolic sine (an odd function) divided by the denominator (an even function) is zero. Then we split the integral in two integrals and find its value:

$$= \frac{15}{2} e_1 \int_{-\log 2}^{\log 2} \frac{dt}{17 - 8 \cosh t} + e_1 \int_{-\log 2}^{\log 2} \frac{dt}{2} = \frac{e_1}{2} \left[\log \frac{-8 \exp(t) + 2}{-8 \exp(t) + 32} \right]_{-\log 2}^{\log 2} + e_1 \log 2 = e_1 \log 7$$

Now we see that the integral following both paths gives the same result, as indicated by the theorem. In fact, the analytical functions can be integrated directly by using the indefinite integral:

$$\begin{aligned} \int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} &= [\log(z-1)]_{(5,-3)}^{(5,3)} = \left[\frac{1}{2} \log((x-1)^2 - y^2) + e_1 \arg \operatorname{tgh} \left(\frac{y}{x-1} \right) \right]_{(5,-3)}^{(5,3)} \\ &= \left[\frac{e_1}{2} \log \left| \frac{x-1+y}{x-1-y} \right| \right]_{(5,-3)}^{(5,3)} = e_1 \log 7 \end{aligned}$$

Analyticity and square of convergence of the power series

A matrix function $f(A)$ can be developed as a Taylor series of powers of the matrix A :

$$f(A) = \sum_{n=0}^{\infty} a_n A^n \quad A = x + y e_1 = \begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix}$$

The series is convergent when all the eigenvalues of the matrix A are located within the radius r of convergence of the complex series:

$$\left| \sum_{n=0}^{\infty} a_n z^n \right| < \infty \quad \text{for} \quad |z| < r$$

which leads us to the following conditions:

$$|x+y| < r \quad \text{and} \quad |x-y| < r$$

Therefore, the region of convergence of a Taylor series of hyperbolic variable is a square centred at the origin of coordinates with vertices $(r, 0)$ – $(0, r)$ – $(-r, 0)$ – $(0, -r)$. Note that from both conditions we obtain:

$$|\det(x + y e_1)| = |x^2 - y^2| < r^2.$$

which is a condition similar to that for complex numbers, that is, the modulus must be lower than the radius of convergence. However this condition is not enough to ensure the convergence of the series. Let us see, for example, the function $f(z) = 1/(1-z)$:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

The radius of convergence of the complex series is $r = 1$, so that the square of the convergence of the hyperbolic series is $(1, 0)$ – $(0, 1)$ – $(-1, 0)$ – $(0, -1)$. Otherwise, we have formerly seen that $1/(z-1)$ (and hence $f(z)$) is not analytic at the lines $x+y-1=0$ and $x-y-1=0$ belonging to the boundary of the square of convergence. Now we find a phenomenon which also happens within the complex numbers: at some point on the boundary of the region of convergence of the Taylor series, the function is not analytic.

Another example is the Riemann's function:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z \log n}$$

Now, we take z instead of real being a hyperbolic number:

$$\zeta(x + y e_1) = \sum_{n=1}^{\infty} e^{-x \log n} (\cosh(y \log n) - e_1 \sinh(y \log n))$$

$$= \sum_{n=1}^{\infty} n^{-x} \left(\frac{n^y + n^{-y}}{2} - \frac{n^y - n^{-y}}{2} e_1 \right) = \sum_{n=1}^{\infty} \left(n^{-x+y} \frac{1-e_1}{2} + n^{-x-y} \frac{1+e_1}{2} \right)$$

The series is convergent only if $-x+y < -1$ and $-x-y < -1$ or:

$$1-x < y < x-1$$

which is the positive half-sector beginning at $(x, y) = (1, 0)$. You can see that it is an analytic function within this domain. We can rewrite the Riemann's function in the form:

$$\zeta(x+y e_1) = \frac{1+e_1}{2} \zeta(x+y) + \frac{1-e_1}{2} \zeta(x-y)$$

and then define the Riemann's zeta function extended to the other sector taking into account the complex analytic continuation when needed for $\zeta(x+y)$ or $\zeta(x-y)$:

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \zeta(z) \Gamma(z) \cos \frac{\pi z}{2} \quad \text{and} \quad \zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt \quad 0 < \text{Re}(z) < 1$$

Now a very interesting theorem arises: every analytic function $f(z)$ can be always written in the following form:

$$f(x+y e_1) = \frac{1+e_1}{2} f(x+y) + \frac{1-e_1}{2} f(x-y)$$

Let us prove this statement calculating the partial derivatives of each part a and b of the analytic function:

$$\frac{\partial a}{\partial x} = \frac{1}{2} \left[\frac{df(x+y)}{d(x+y)} + \frac{df(x-y)}{d(x-y)} \right] \quad \frac{\partial b}{\partial x} = \frac{1}{2} \left[\frac{df(x+y)}{d(x+y)} - \frac{df(x-y)}{d(x-y)} \right]$$

$$\frac{\partial a}{\partial y} = \frac{1}{2} \left[\frac{df(x+y)}{d(x+y)} - \frac{df(x-y)}{d(x-y)} \right] \quad \frac{\partial b}{\partial y} = \frac{1}{2} \left[\frac{df(x+y)}{d(x+y)} + \frac{df(x-y)}{d(x-y)} \right]$$

These derivatives fulfil the analyticity condition provided that the derivatives of the real function at $x+y$ and $x-y$ exist. On the other hand, this expression is a method to get the analytical continuation of any real function. For example, let us construct the analytical continuation of $f(x) = \cos x$:

$$\begin{aligned} \cos(x+y e_1) &= \frac{1+e_1}{2} \cos(x+y) + \frac{1-e_1}{2} \cos(x-y) \\ &= \frac{1}{2} [\cos(x+y) + \cos(x-y)] + \frac{e_1}{2} [\cos(x+y) - \cos(x-y)] \\ &= \cos x \cos y - e_1 \sin x \sin y \end{aligned}$$

The other consequence of this way of analytic continuation is the fact that if $f(z)$ loses analyticity for the real value x_0 then it is neither analytic at the lines with slope 1 and -1 passing through $(x_0, 0)$. Since we have supposed that the function is analytic except for a certain real value, the former equality holds although the analyticity is lost at this point. But then the function cannot be analytic at $(x_0 + t, -t)$ nor $(x_0 - t, t)$ because:

$$f(x_0 + t - t e_1) = \frac{1 + e_1}{2} f(x_0 + 2t) + \frac{1 - e_1}{2} f(x_0)$$

$$f(x_0 - t + t e_1) = \frac{1 + e_1}{2} f(x_0) + \frac{1 - e_1}{2} f(x_0 - 2t)$$

For example, look at the function $f(z) = 1/z$, which is not analytic for the real value $x = 0$. Then it will not be analytic at the lines $y = x$ and $y = -x$. We can see it through the decomposition of the function:

$$\frac{1}{z} = \frac{1}{x + y e_1} = \frac{x - y e_1}{x^2 - y^2} = \frac{1 + e_1}{2} \frac{1}{x + y} + \frac{1 - e_1}{2} \frac{1}{x - y}$$

Now we may return to the question of the convergence of the Taylor series. A function can be only developed in a power series in the neighbourhood of a point where it is analytic. The series is convergent till where the function breaks the analyticity, so that the lines $y = x - x_0$ and $y = -x + x_0$ (being x_0 the real value for which $f(x)$ is not analytic) are boundaries of the convergence domain. On the other hand, due to the symmetry of the powers of z , the convergence domain forms a square around the centre of development. This implies that there are not multiply connected domains. The Gruyère cheese picture of a complex domain is not possible within the hyperbolic numbers, because if the function is not analytic at a certain point, then it is not analytic for all the points lying on a cross which passes through this point. The hyperbolic domains are multiply separated, in other words, formed by disjointed rectangular regions without holes.

About the isomorphism of Clifford algebras

Until now, I have only used the geometric algebra generated by the Euclidean plane vectors (usually noted as $Cl_{2,0}(\mathbf{R})$). This algebra already contains the hyperbolic numbers and hyperbolic vectors. However an isomorphic algebra $Cl_{1,1}(\mathbf{R})$ generated by hyperbolic vectors (time-space) is more used in relativity:

$$e_0^2 = 1 \quad e_1^2 = -1 \quad e_0 e_1 = -e_1 e_0 \quad (e_0 e_1)^2 = 1$$

The isomorphism is:

$$Cl_{2,0}(\mathbf{R}) \leftrightarrow Cl_{1,1}(\mathbf{R})$$

$$\begin{array}{lll} 1 & \leftrightarrow & 1 \\ e_1 & \leftrightarrow & e_{01} \\ e_2 & \leftrightarrow & e_0 \\ -e_{12} & \leftrightarrow & e_1 \end{array}$$

A hyperbolic number has the expression $z = x + y e_{01}$ and a hyperbolic vector $v = v_0 e_0 + v_1 e_1$.

This description perhaps could be satisfactory for a mathematician or a physicist, but not for me. I think that in fact both algebras only differ in the notation but not in their nature, so that they are the same algebra. Moreover, they are equal to the matrix algebra, which is the expected algebra for a space of multiple quantities (usually said *vectors*). The plane geometric algebra is the algebra of the 2×2 real matrices⁵.

$$Cl_{2,0}(\mathbf{R}) = Cl_{1,1}(\mathbf{R}) = \mathbf{M}_{2 \times 2}(\mathbf{R})$$

This distinct notation only expresses the fact that the plane geometric algebra is equally generated (in the Grassmann's sense) by Euclidean vectors or hyperbolic vectors. The vector plane (Euclidean or hyperbolic) is the quotient space of the geometric algebra divided by an even subalgebra (complex or hyperbolic numbers). Just this is the matter of the next chapter.

Exercises

12.1 Calculate the square roots of $5 + 4 e_1$.

12.2 Solve the following equation: $2 z^2 + 3 z - 17 + 3 e_1 = 0$

12.3 Solve directly the equation: $z^2 - 6 z + 5 = 0$

12.4 Find the analytical continuation of the function $f(x) = \sin x$.

12.5 Find $\cosh 4\psi$ and $\sinh 4\psi$ as functions of $\cosh \psi$ and $\sinh \psi$.

12.6 Construct the analytical continuation of the real logarithm and see that it is identical to the logarithm found from the hyperbolic exponential function.

12.7 Calculate the integral $\int_{-e_1}^{e_1} z^2 dz$ following a straight path $z = t e_1$ and a circular path $z = \cos t + e_1 \sin t$, and see that the result is identical to the integration via the primitive.

12.8 Prove that if $f(z)$ is a hyperbolic analytic function and does not vanish then it is a hyperbolic conformal mapping.

⁵ In comparison, the 2×2 complex matrices (Pauli matrices) are a representation of the algebra of the tridimensional space.

13. THE HYPERBOLIC OR PSEUDO-EUCLIDEAN PLANE

Hyperbolic vectors

The elements not belonging to the subalgebra of the hyperbolic numbers form a quotient space¹, but not an algebra:

$$v = v_2 e_2 + v_{21} e_{21} = \begin{pmatrix} 0 & v_2 - v_{21} \\ v_2 + v_{21} & 0 \end{pmatrix}$$

In other words, their linear combinations are in the space, but the products are hyperbolic numbers. These elements play the same role as the Euclidean vectors with respect to the complex numbers. Because of this, they are called *hyperbolic vectors*. Like the hyperbolic numbers, the hyperbolic vectors have also a pseudo-Euclidean determinant:

$$\det(v_2 e_2 + v_{21} e_{21}) = v_{21}^2 - v_2^2$$

Following the analogy with Euclidean vectors:

$$\det(w_1 e_1 + w_2 e_2) = -(w_1 e_1 + w_2 e_2)^2 = -|w|^2$$

the determinant of the hyperbolic vectors is also equal to its square with opposite sign:

$$\det(v_2 e_2 + v_{21} e_{21}) = -(v_2 e_2 + v_{21} e_{21})^2 = -|v|^2$$

whence the modulus of a hyperbolic vector can be defined:

$$|v| = \sqrt{v_2^2 - v_{21}^2}$$

Like for Euclidean vectors, the inverse of a hyperbolic vector is equal to this vector divided by its square (or square of the modulus):

$$(v_2 e_2 + v_{21} e_{21})^{-1} = \frac{v_2 e_2 + v_{21} e_{21}}{v_2^2 - v_{21}^2}$$

As before, there is not any privileged direction in the plane. Then every subspace whose elements have the form:

¹ Every Euclidean vector is obtained from e_1 through a rotation and dilation given by the complex number with the same components: $v_1 e_1 + v_2 e_2 = e_1 (v_1 + v_2 e_{12})$ so that the complex and vector planes are equivalent. This statement also holds in hyperbolic geometry: every hyperbolic vector is obtained from e_2 through a rotation and dilation given by the hyperbolic number with the same components: $v_2 e_2 + v_{21} e_{21} = e_2 (v_2 + v_{21} e_1)$ or in relativistic notation: $v_0 e_0 + v_1 e_1 = e_0 (v_0 + v_1 e_{01})$ so that both hyperbolic planes are equivalent. That reader accustomed to the relativistic notation should remember the isomorphism: $e_0 \leftrightarrow e_2$, $e_1 \leftrightarrow e_{21}$, $e_{01} \leftrightarrow e_1$.

$$v_w w + v_{21} e_{21}$$

where w is a unitary Euclidean vector perpendicular to the unitary Euclidean vector u , is also a subspace of hyperbolic vectors complementary of the hyperbolic numbers with the direction of u .

The hyperbolic vectors have the following properties, which you may prove:

- 1) The product of two hyperbolic vectors is always a hyperbolic number. This fact is shown by the following table, where the products of all the elements of the geometric algebra are summarised:

$\times$	hyp. numbers	hyp. vectors
hyp. numbers	hyp. numbers	hyp. vectors
hyp. vectors	hyp. vectors	hyp. numbers

- 2) The conjugate of a product of two hyperbolic vectors x and y is equal to the product with these vectors exchanged:

$$(x y)^* = y x$$

- 3) The product of three hyperbolic vectors x , y and z fulfils the permutative property:

$$x y z = z y x$$

When a hyperbolic number n and a hyperbolic vector x are exchanged, the hyperbolic number becomes conjugated according to the permutative property:

$$n x = x n^*$$

- 4) The modulus of a product of hyperbolic vectors is equal to the product of moduli:

$$|v w| = |v| |w|$$

This property follows immediately from the fact that the determinant of a product of matrices is equal to the product of determinants of each matrix.

Inner and outer products of hyperbolic vectors

For any two hyperbolic vectors r and s such as:

$$r = r_2 e_2 + r_{21} e_{21} \quad s = s_2 e_2 + s_{21} e_{21}$$

their inner and outer products are defined by means of the geometric (or matrix) product in the following way:

$$r \cdot s = \frac{r s + s r}{2} = r_2 s_2 - r_{21} s_{21}$$

$$r \wedge s = \frac{r s - s r}{2} = (r_2 s_{21} - r_{21} s_2) e_1$$

Then the product of two vectors can be written as the addition of both products:

$$r s = r \cdot s + r \wedge s$$

If the outer product of two hyperbolic vectors vanishes, they are proportional:

$$r \wedge s = 0 \quad \Leftrightarrow \quad \frac{r_2}{s_2} = \frac{r_{12}}{s_{12}} \quad \Leftrightarrow \quad r \parallel s$$

Two hyperbolic vectors are said to be *orthogonal* if their inner product vanishes:

$$r \perp s \quad \Leftrightarrow \quad r \cdot s = 0 \quad \Leftrightarrow \quad r_2 s_2 - r_{21} s_{21} = 0 \quad \Leftrightarrow \quad \frac{r_2}{r_{21}} = \frac{s_{21}}{s_2}$$

When the last condition is fulfilled, we see both vectors being symmetric with respect to any quadrant bisector. When the ratio of components is equal to ± 1 (directions of quadrant bisectors), the vectors have null modulus and are self-orthogonal. For ratios differing from ± 1 , one vector has negative determinant and the other positive determinant, so that they belong to different sectors. That is, there is not any pair of orthogonal vectors within the same sector.

Angles between hyperbolic vectors

The oriented angle α between two Euclidean vectors u and v (of the form $x e_1 + y e_2$) is obtained from the complex exponential function:

$$\exp(\alpha e_{12}) = \frac{u v}{|u| |v|} \quad \Leftrightarrow \quad \begin{cases} \cos \alpha = \frac{u \cdot v}{|u| |v|} \\ \sin \alpha = \frac{u \wedge v e_{21}}{|u| |v|} \end{cases} \quad \Leftrightarrow \quad \alpha = \arctg \frac{u \wedge v e_{21}}{u \cdot v} + (\pi e_{12})$$

Then, the oriented hyperbolic angle ψ between two hyperbolic vectors u and v (of the form $x e_2 + y e_{21}$) is defined as:

$$\exp(\psi e_1) = \frac{u v}{|u| |v|} \quad \Leftrightarrow \quad \begin{cases} \cosh \psi = \frac{u \cdot v}{|u| |v|} \\ \sinh \psi = \frac{u \wedge v e_1}{|u| |v|} \end{cases} \quad \Leftrightarrow$$

$$\psi = \arg \operatorname{tgh} \left(\frac{u \wedge v e_1}{u \cdot v} \right) + (\pi e_{12}) = \frac{1}{2} \log \left(\frac{u \cdot v + u \wedge v e_1}{u \cdot v - u \wedge v e_1} \right) + (\pi e_{12})$$

The parenthesis for πe_{12} indicates that this angle is added to the arctangent or not depending on the quadrant (Euclidean plane) or half sector (hyperbolic plane).

Let us analyse whether these expressions are suitable. The hyperbolic cosine is always greater or equal to 1. It cannot describe the angle between two vectors belonging to opposite half sectors, which have a negative inner product. So we must keep the complex analytic continuation of the hyperbolic sine and cosine:

$$\sinh(x + e_{12} y) \equiv \sinh x \cos y + e_{12} \cosh x \sin y$$

$$\cosh(x + e_{12} y) \equiv \cosh x \cos y + e_{12} \sinh x \sin y$$

to get $\sinh(\psi + e_{12} \pi) \equiv -\sinh \psi$ and $\cosh(\psi + e_{12} \pi) \equiv -\cosh \psi$ with ψ real. Therefore opposite hyperbolic vectors form a circular angle of π radians².

Let us consider the angle between vectors lying on different sectors. The modulus of one vector is real while that of the other is imaginary what implies that the hyperbolic sine and cosine of the angle are imaginary. Using again the complex analytic continuation of the hyperbolic sine and cosine we find these imaginary values:

$$\sinh\left(\psi \pm e_{12} \frac{\pi}{2}\right) \equiv \pm e_{12} \cosh \psi \quad \psi \text{ real}$$

$$\cosh\left(\psi \pm e_{12} \frac{\pi}{2}\right) \equiv \pm e_{12} \sinh \psi$$

Then, which is the angle between two orthogonal vectors? Since they have a null inner product while their outer product equals in real value the product of both moduli and the hyperbolic cosine is $\pm e_{12}$, they form an angle of $\pi e_{12} / 2$ or $3\pi e_{12} / 2$, that is, orthogonal hyperbolic vectors form a circular right angle:

$$\psi_{\perp} = \frac{1}{2} \log(-1) + (\pi e_{12}) = \frac{\pi}{2} e_{12} + (\pi e_{12})$$

The analytic continuations of the hyperbolic trigonometric functions must be consistent with their definitions given in the page 142. For example, in the lowest half

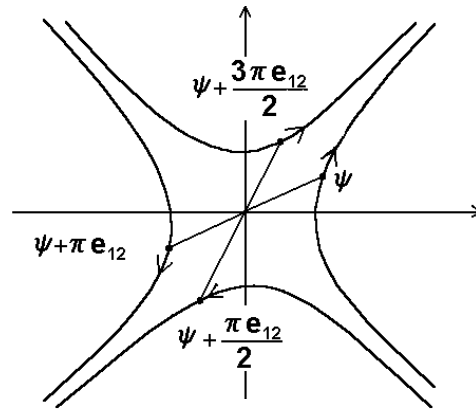


Figure 13.1

² The so called “antimatter” is not really any special kind of particles, but matter with the energy-momentum vector (E, p) lying on the negative half sector instead of the positive one (usual matter). Also we may wonder whether matter having (E, p) on the imaginary sector exists since, from a geometric point of view, there is not any obstacle. It is known that the particle dynamics for $c < V < \infty$ is completely symmetric to the dynamics for $0 \leq V < c$, but with an imaginary mass. In this case, the head of the energy-momentum vector is always located on the hyperbola having the y -axis (p axis) as principal axis of symmetry. According to Einstein’s formula, for $V = \infty$ we have $E = 0$ and $p = m c$, that is, the particle has a null energy! The technical question is how can a particle trespass the light barrier?

sector, the modulus of the hyperbolic radius is imaginary. The abscissa x and ordinate y divided by r , the modulus of the radius vector, is also imaginary so the angle is a real quantity plus a right angle and the hyperbolic sine and cosine are exchanged. The values of the angles consistent with the analytic continuation are displayed in the figure 13.1.

Congruence of segments and angles

Two segments (vectors) are *congruent* (have *equal length*) if their determinants are equal, in other words, when one segment can be obtained from the other through an isometry. Segments having equal length always lie on the same sector.

Two hyperbolic angles are said to be *congruent* (or *equal*) if they have the same moduli, that is, if they intercept arcs of hyperbola with the same hyperbolic length.

A triangle is said to be *isosceles* if it has two congruent sides. In colloquial language we talk about “sides with equal length”, even “equal sides”. Let us see the *isosceles triangle theorem*: the angles adjacent to the base of an isosceles triangle are equal. The proof uses the outer product. The triangle area is the half of the outer product of any two sides, as proved below in the section on the area. Suppose that the sides a and b have the same modulus. Then:

$$|a| |c| |\sinh \beta| = |b| |c| |\sinh \alpha| \quad \Rightarrow \quad |\beta| = |\alpha|$$

Isometries

In the Euclidean plane, an isometry is a geometric transformation that preserves the modulus of vectors and complex numbers. Now I give a more general definition: a geometric transformation is an *isometry* if it preserves the determinant of every element of the geometric algebra. In fact, the isometries are the inner automorphism of matrices:

$$\mathbf{A}' = \mathbf{B}^{-1} \mathbf{A} \mathbf{B} \Rightarrow \det \mathbf{A}' = \det \mathbf{A}$$

When $\mathbf{B}$ represents a Euclidean vector, the isometry is a reflection in the direction of $\mathbf{B}$. If a complex number of argument α is represented by $\mathbf{B}$, the transformation is a rotation of angle 2α .

We wish now to obtain the isometries for hyperbolic vectors. The hyperbolic rotation of a hyperbolic vector (figure 13.2) is obtained through the product by an exponential having unity determinant:

$$v'_2 e_2 + v'_{21} e_{21} = (v_2 e_2 + v_{21} e_{21})(\cosh \psi + \sinh \psi e_1)$$

Writing the components, we have:

$$\begin{cases} v'_2 = v_2 \cosh \psi + v_{21} \sinh \psi \\ v'_{21} = v_2 \sinh \psi + v_{21} \cosh \psi \end{cases}$$

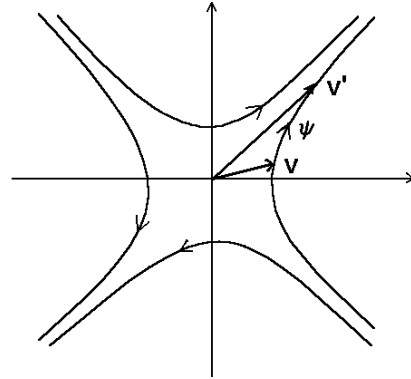


Figure 13.2

which is the Lorentz transformation³ of the relativity. When a vector is turned through a positive angle ψ , its extreme follows the hyperbola in the direction shown by the arrowheads in the figure 13.2, as is deduced from the components. So, they indicate the geometric positive sense of hyperbolic angles (a not trivial question). Since the points of intersections with the axis corresponds to $\psi = 0$ plus multiples of $\pi e_{12}/2$, the signs of the hyperbolic angles are determined: positive in the first and third quadrant, and negative in the second and fourth quadrant.

We can also write the hyperbolic rotation as an inner automorphism of matrices by using the half argument identity for hyperbolic trigonometric functions:

$$\cosh \psi + e_1 \sinh \psi \equiv \left(\cosh \frac{\psi}{2} + e_1 \sinh \frac{\psi}{2} \right)^2$$

and the permutative property:

$$v'_2 e_2 + v'_{21} e_{21} = \left(\cosh \frac{\psi}{2} - \sinh \frac{\psi}{2} e_1 \right) (v_2 e_2 + v_{21} e_{21}) \left(\cosh \frac{\psi}{2} + \sinh \frac{\psi}{2} e_1 \right)$$

If $z = \exp(e_1 \psi/2)$, then the rotation is written as:

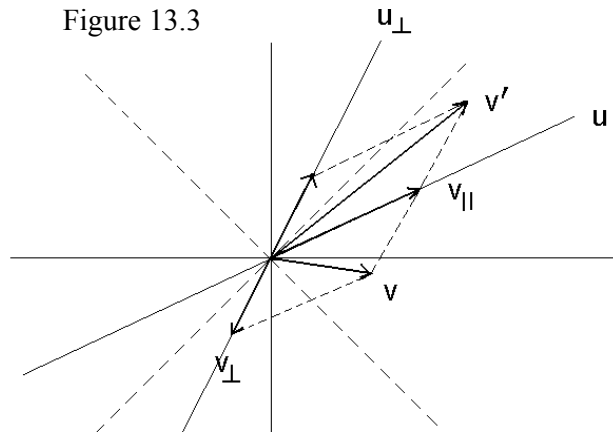
$$v' = z^{-1} v z$$

Now it is not needed that z be unitary and can have any modulus. Also, we see that the hyperbolic rotation leaves the hyperbolic numbers invariant.

Analogously to reflections in the Euclidean plane, the hyperbolic reflection of a hyperbolic vector v with respect to a direction given by the vector u is:

$$v' = u^{-1} v u = u^{-1} (v_{\parallel} + v_{\perp}) u = v_{\parallel} - v_{\perp}$$

because the proportional vectors commute and those which are orthogonal anticommute (also in the hyperbolic case, because the inner product is zero). Every pair of orthogonal directions (for example $u_{\parallel}$ and $u_{\perp}$ in figure 13.3) are always seen by us as having symmetry with respect to the quadrant bisectors. Since the determinant is preserved, a vector and its reflection always belong to the same sector, the hyperbolic angle between each vector and the direction of reflection being equal with opposite sign.



³ The hyperbolic vector in the space-time is $ct e_0 + x e_1$ where x is the space coordinate, t the time and c the light celerity. The argument of the hyperbolic rotation ψ is related with the relative velocity V of both frames of reference through $\psi = \arg \tanh V/c$.

Finally, note that the extremes of a vector and its transformed vector under a hyperbolic reflection (or any isometry) lie on an equilateral hyperbola.

Theorems about angles

The sum of the oriented angles of a hyperbolic triangle is minus two right circular angles. To prove this theorem, draw a line parallel to a side and apply the Z-theorem (figure 13.4): they will form an angle between opposite directions, which has the value $-\pi e_{12}$. The minus sign is caused by the fact that, following the positive orientation of angles, a right angle is subtracted each time an asymptote is trespassed.

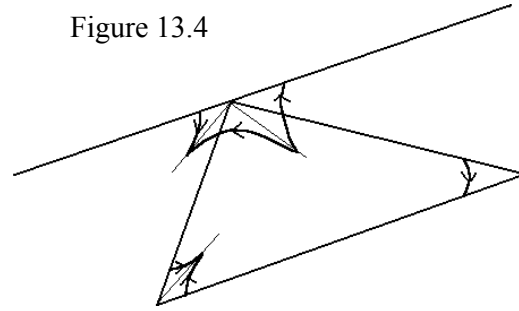


Figure 13.4

Look at the figure 13.5 where an angle inscribed in an equilateral hyperbola is drawn. A diameter divides the inscribed angle into the angles α and β . Then we draw two radius from the origin to both extremes of the angle. All the radius have the same length, so that the upper and lowest triangle are isosceles and have two equal angles. Since the sum of the angles of each hyperbolic triangle is $-\pi e_{12}$, the third angle is $-\pi e_{12}-2\alpha$ and $-\pi e_{12}-2\beta$ respectively. So the supplementary angles are 2α and 2β , and hence the theorem follows: the inscribed angle on a hyperbola $x^2 - y^2 = r^2$ is the half of the central angle (intercepted arc) so that it is constant and independent of the location of its vertex.

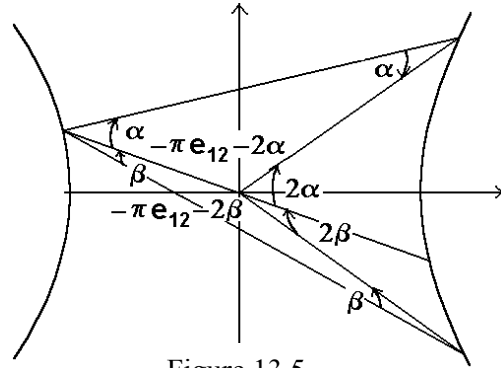


Figure 13.5

For example, take the points (5, 3) and (5, -3) as extremes of an inscribed angle with the vertex placed at the negative branch, for example at (-4, 0) or (-5, -3). Even you can take the vertex at the positive branch, e. g. at (4, 0). Anyway its value is $\log 2$. Now calculate the central angle or intercepted arc (with vertex at the origin) and see that its value is $\log 4$, twice the inscribed angle.

Distance between points

The *distance between two points* on the hyperbolic plane is defined as the modulus of the vector going from one point to the another:

$$d(P, Q) = |PQ| = \sqrt{(x_Q - x_P)^2 - (y_Q - y_P)^2}$$

In the hyperbolic plane, the distance is a real or imaginary positive number (depending on the sector), which has the following properties:

- 1) $d(P, Q) = d(Q, P)$
- 2) If the vectors PQ , QR and PR lie on the positive half sector then they fulfil the triangular inequality:

$$d(P, R) \geq d(P, Q) + d(Q, R)$$

The prove is obtained by means of the inner product:

$$\begin{aligned} PR^2 &= (PQ + QR)^2 = PQ^2 + QR^2 + 2 PQ \cdot QR \\ &= PQ^2 + QR^2 + 2 |PQ| |QR| \cosh \psi \geq PQ^2 + QR^2 + 2 |PQ| |QR| \end{aligned}$$

The extraction of the square root yields the triangular inequality:

$$|PR| \geq |PQ| + |QR|$$

So the sum of two sides of a triangle is smaller than the third side whenever they are taken in the positive half sector. For example, let us apply this statement to the triangle with vertices $A = (5, 3)$, $B = (1, 0)$ and $C = (10, 1)$. Taking the modulus of the sides positive, we have:

$$\begin{aligned} |BC| &= \sqrt{80} & |BA| &= \sqrt{7} & |AC| &= \sqrt{21} \\ |BC| &\geq |BA| + |AC| & \Rightarrow & \sqrt{80} \geq \sqrt{7} + \sqrt{21} \end{aligned}$$

This result must be commented with more detail. Firstly, we see that the straight path has the highest length among the possible paths between two given points. In the Minkowski's space-time this fact causes the *twin paradox*. Since the pseudo-Euclidean length of the path is the proper time for each person, the brother who followed the straight path -going in an inertial frame- has aged more than the brother who followed another path -subjected to accelerations-.

Area on the hyperbolic plane

Now I give a general geometric definition of area valid for both Euclidean and hyperbolic planes. If a parallelogram has orthogonal sides, then the modulus of its area is equal to the product of the lengths of the orthogonal sides. In Euclidean geometry a parallelogram with orthogonal sides is called a *rectangle*. In the hyperbolic plane, two sides are orthogonal if their directions are seen by us as symmetric with respect to the direction of the quadrant bisector. So, I have preferred to avoid the word *rectangle* in this case, while the term *parallelogram* is still valid within this context.

Which is the suitable algebraic expression to calculate the area of any parallelogram? Suppose that it is the outer product. If a and b are the sides of the parallelogram then:

$$|A| = |a \wedge b| = |a| |b| |\sinh \psi(a, b)|$$

Let us see the consistence of this expression. First at all, recall that the angle between orthogonal hyperbolic vectors is a right circular angle:

$$\psi_{\perp} = \pm \frac{\pi}{2} e_{12}$$

so that the area of a parallelogram with orthogonal sides is the product of their lengths as expected:

$$|A| = |a| |b| \left| \sinh \left(\pm \frac{\pi}{2} e_{12} \right) \right| = |a| |b| \sin \frac{\pi}{2} = |a| |b|$$

For any parallelogram not having orthogonal sides, the area is the product of the *base* (one side) for the *altitude* (the projection of the other side onto the direction orthogonal to the base) and this product is only given by the outer product because its anticommutativity removes the proportional projection:

$$|A| = |a \wedge b| = |(a_{\parallel} + a_{\perp}) \wedge b| = |a_{\perp}| |b|$$

The expression for the area in Cartesian components is equal to that for the Euclidean plane. It means that we can calculate areas graphically in the usual way, fact that has allowed till now to define the hyperbolic trigonometric functions from the area scanned by the hyperbola radius in the Euclidean plane, in spite of not being their proper plane. However, as I have explained in the footnote at the page 142, we must perceive the pseudo-Euclidean nature of the area in the hyperbolic plane. The figure 13.6

displays how are the radii perpendicular to the hyperbola because the bisector of the radius and the tangent vector is parallel to the quadrant bisector. Also it may be proved analytically by means of differentiation of the equation $x^2 - y^2 = r^2$:

$$2x dx - 2y dy = 0 \quad \Leftrightarrow \quad r \cdot dr = 0$$

showing that the radius vector and its differential are orthogonal for the hyperbola with constant radius. On the other hand, observe that the triangles drawn in the figure 13.6 are isosceles and have two equal angles approaching the right angle value at the limit of null area.

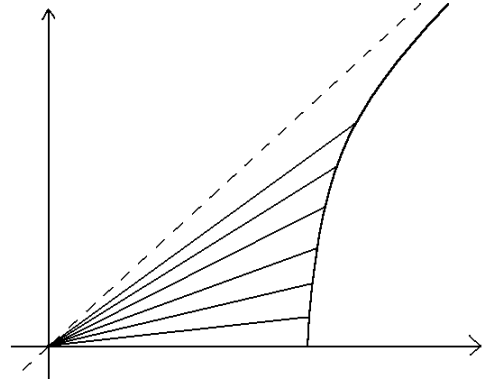


Figure 13.6

Diameters of the hyperbola and Apollonius' theorem

At the page 129, I postponed an interpretation of the central equation for the hyperbola analogous to the cylindrical angles for the ellipse because it is properly of pseudo-Euclidean nature. Here I develop this interpretation.

If P is any point on a hyperbola with major and minor half-axis OQ and OS (figure 11.17 reproduced below) then its central equation is:

$$OP = \pm \frac{OQ \cosh \chi + OS \frac{\sinh \chi}{\sqrt{e^2 - 1}}}{\sqrt{\cosh^2 \chi - \frac{\sinh^2 \chi}{e^2 - 1}}}$$

Since OQ and OS are orthogonal, the square of this equation fulfils:

$$\frac{OP^2}{OQ^2} \cosh^2 \chi + \frac{OP^2}{OS^2} \sinh^2 \chi = 1$$

Both half-axis are related with the eccentricity through:

$$OS^2 = (1 - e^2) OQ^2$$

Note that $OS^2 < 0$ since it is the square of an ordinate on the hyperbolic plane.

Introducing the hyperbolic angle ψ in the following way:

$$\cosh \psi = \frac{\cosh \chi}{\sqrt{\cosh^2 \chi - \frac{\sinh^2 \chi}{e^2 - 1}}}$$

now the equation of the hyperbola becomes:

$$OP = \pm (OQ \cosh \psi + OS \sinh \psi)$$

Observe in the figure 13.7 that any hyperbola can be obtained as an intersection of a transverse plane with the equilateral hyperbolic prism. The plane of the acute hyperbola (with $1 < e < \sqrt{2}$) forms an angle ϕ with respect to the horizontal, and hence:

$$\cosh \psi = \frac{\cosh \chi \cos \phi}{\sqrt{\cosh^2 \chi \cos^2 \phi - \sinh^2 \chi}}$$

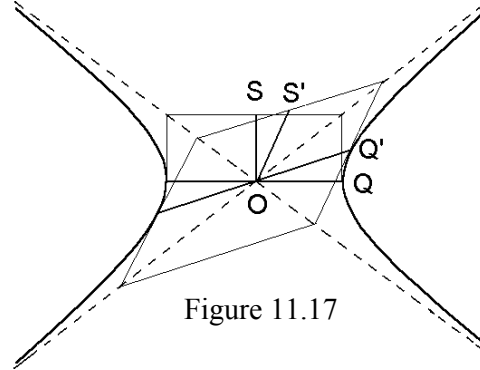


Figure 11.17

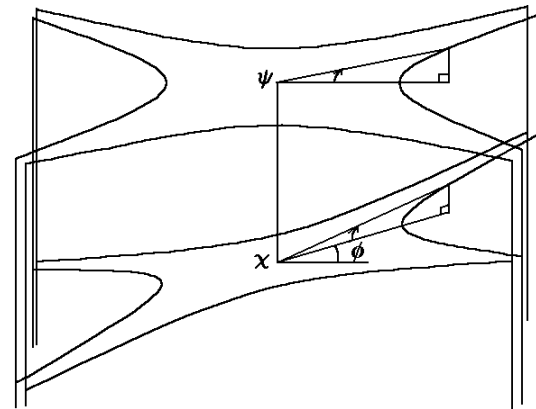


Figure 13.7

So the eccentricity e of the hyperbola is related with the obliquity ϕ of the transverse plane through the relationship:

$$\cos \phi = \sqrt{e^2 - 1} \quad \text{with} \quad 1 < e < \sqrt{2}$$

The Apollonius' conjugate diameters of any hyperbola are the intersections of the transverse plane with a pair of orthogonal axial planes; in other words, two radii are conjugate (figure 11.17) if their projections onto the horizontal plane are turned through the same hyperbolic angle φ :

$$OQ' = OQ \cosh \varphi + OS \sinh \varphi$$

$$OS' = OQ \sinh \varphi + OS \cosh \varphi$$

Our Euclidean eyes see the horizontal projections as symmetric lines with respect to the quadrant bisector. However, they are actually orthogonal because:

$$OQ'^2 - OS'^2 = OQ^2 - OS^2$$

and, therefore, can be taken as a new system of orthogonal coordinates. Even we can draw a new picture with the new diameters on the Cartesian axis.

The central equation of the hyperbola using the rotated axis is:

$$OP = \pm(OQ' \cosh(\psi - \varphi) + OS' \sinh(\psi - \varphi))$$

which shows that a hyperbolic rotation of the coordinate axis has been made with respect to the principal diameter of the hyperbola.

The law of sines and cosines

Since the modulus of the area is identical on the Euclidean and hyperbolic planes, a parallelogram is divided by its diagonal in two triangles of equal area. This statement is somewhat subtle since the Euclidean congruence of triangles is not valid in the hyperbolic plane. I shall return to this question later. Now we only need to know that the area of a hyperbolic triangle is the half of the outer product of any two sides.

Following the perimeter of a triangle, let a , b , and c be its sides respectively opposite to the angles α , β and γ . Then the angles formed by the oriented sides are supplementary of the angles of the triangle and:

$$a \wedge b = b \wedge c = c \wedge a \Rightarrow -|a||b| \sinh \gamma = -|b||c| \sinh \alpha = -|c||a| \sinh \beta$$

$$\frac{|a|}{\sinh \alpha} = \frac{|b|}{\sinh \beta} = \frac{|c|}{\sinh \gamma}$$

which is the *law of sines*.

From $a + b + c = 0$, we have:

$$a^2 = (-b - c)^2 = b^2 + c^2 + 2b \cdot c \quad \Rightarrow \quad a^2 = b^2 + c^2 - 2|b||c| \cosh \alpha$$

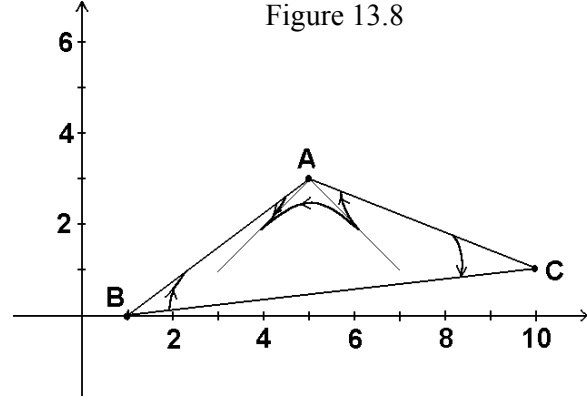
which is the *law of cosines*. And also:

$$b^2 = a^2 + c^2 - 2|a||c| \cosh \beta$$

$$c^2 = a^2 + b^2 - 2|a||b| \cosh \gamma$$

When applying both theorems, we must take care with the sides having imaginary length and the signs of the angles and trigonometric functions.

As an application of the law of sines and cosines, consider the hyperbolic triangle with vertices $A=(5, 3)$, $B=(1, 0)$, $C=(10, 1)$, whose sides belong to the real sector (figure 13.8):



$$AB = B - A = -4e_2 - 3e_{21} \quad |AB| = \sqrt{(-4)^2 - (-3)^2} = \sqrt{7}$$

$$BC = C - B = 9e_2 + e_{21} \quad |BC| = \sqrt{9^2 - 1^2} = \sqrt{80}$$

$$CA = A - C = -5e_2 + 2e_{21} \quad |CA| = \sqrt{(-5)^2 - 2^2} = \sqrt{21}$$

$$BC^2 = CA^2 + AB^2 - 2|CA||AB| \cosh \alpha \quad \cosh \alpha = -\frac{26}{7\sqrt{3}}$$

$$CA^2 = AB^2 + BC^2 - 2|AB||BC| \cosh \beta \quad \cosh \beta = \frac{33}{4\sqrt{35}}$$

$$AB^2 = BC^2 + CA^2 - 2|BC||CA| \cosh \gamma \quad \cosh \gamma = \frac{47}{4\sqrt{105}}$$

From where it follows that:

$$\alpha = -1.3966... - \pi e_{12} \quad \beta = 0.8614... \quad \gamma = 0.5352...$$

I have obtained their signs from the definition of the angles $\alpha = BAC$, $\beta = CBA$, $\gamma = ACB$ and the geometric plot (figure 13.8). Note that $\alpha + \beta + \gamma = -\pi e_{12}$ and they fulfil the law of sines:

$$\frac{|BC|}{\sinh \alpha} = \frac{|CA|}{\sinh \beta} = \frac{|AB|}{\sinh \gamma}$$

because $\sinh \alpha = \sinh(-1.3966 + \pi e_{12}) = -\sinh(-1.3966) = \sinh 1.3966$.

Consider now another triangle $A = (2, 4)$, $B = (1, 0)$ and $C = (6, 1)$, having sides on real and imaginary sectors (figure 13.9):

$$AB = B - A = -e_2 - 4e_{21}$$

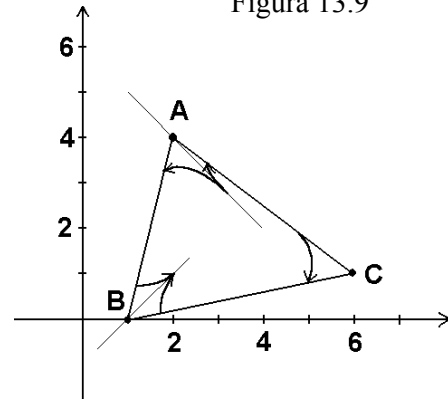
$$|AB| = \sqrt{(-1)^2 - (-4)^2} = \sqrt{15} e_{12}$$

$$BC = C - B = 5e_2 + e_{21}$$

$$|BC| = \sqrt{5^2 - 1^2} = \sqrt{24}$$

$$CA = A - C = -4e_2 + 3e_{21}$$

$$|CA| = \sqrt{(-4)^2 - 3^2} = \sqrt{7}$$



$$BC^2 = CA^2 + AB^2 - 2|CA||AB|\cosh \alpha \quad \cosh \alpha = \frac{16e_{12}}{\sqrt{105}}$$

$$CA^2 = AB^2 + BC^2 - 2|AB||BC|\cosh \beta \quad \cosh \beta = \frac{-e_{12}}{6\sqrt{10}}$$

$$AB^2 = BC^2 + CA^2 - 2|BC||CA|\cosh \gamma \quad \cosh \gamma = \frac{23}{2\sqrt{42}}$$

From the last hyperbolic cosine, which is real, we find the hyperbolic sine for γ , which is a positive angle as shown by the plot:

$$\sinh \gamma = \frac{19}{2\sqrt{42}}$$

which we may use in the law of sines:

$$\frac{\sqrt{24}}{\sinh \alpha} = \frac{\sqrt{7}}{\sinh \beta} = \frac{\sqrt{15}e_{12}}{19/2\sqrt{42}} \quad \Rightarrow \quad \sinh \alpha = -\frac{19e_{12}}{\sqrt{105}} \quad \sinh \beta = -\frac{19e_{12}}{6\sqrt{10}}$$

Recalling that for ψ real:

$$\sinh\left(\psi \pm e_{12} \frac{\pi}{2}\right) \equiv \pm e_{12} \cosh \psi \quad \cosh\left(\psi \pm e_{12} \frac{\pi}{2}\right) \equiv \pm e_{12} \sinh \psi$$

the angles α , β and γ follow:

$$\alpha = -1.2284... - \frac{\pi}{2} e_{12} \quad \beta = 0.0527... - \frac{\pi}{2} e_{12} \quad \gamma = 1.1757$$

Observe that the addition of the three angles is $-\pi e_{12}$, as expected. According to the definition $\alpha = BAC$, $\beta = CBA$, $\gamma = ACB$, let you see the consistence with the geometric plot. The angle γ has positive sign as shown by the figure 13.9. The bisector of α parallel to the quadrant bisector divides it in two angles, one real and the other complex (with imaginary part $\pi e_{12}/2$). The algebraic addition of both angles is α . Taking into account that the second has opposite orientation and must be subtracted and predominates over the first, the negative value of α is explained.

Hyperbolic similarity

Two triangles ABC and $A'B'C'$ are said to be *directly similar*⁴ and their vertices and sides denoted with the same letters are *homologous* if:

$$AB BC^{-1} = A'B' B'C'^{-1} \quad \Rightarrow \quad AB^{-1} A'B' = BC^{-1} B'C'$$

One can prove easily that the third quotient of homologous sides also coincides with the other quotients:

$$AB^{-1} A'B' = BC^{-1} B'C' = CA^{-1} C'A' = r$$

The *similarity ratio* r is defined as the quotient of every pair of homologous sides, which is a hyperbolic number. The modulus of the similarity ratio is the size ratio and the argument is the angle of rotation of the triangle $A'B'C'$ with respect to the triangle ABC .

$$r = \frac{|A'B'|}{|AB|} \exp[\alpha(AB, A'B')e_1]$$

The definition of similarity is generalised to any pair of polygons in the following way. The polygons $ABC...Z$ and $A'B'C'...Z'$ are said to be directly similar with similarity ratio r and the sides denoted with the same letters to be homologous if:

$$r = AB^{-1} A'B' = BC^{-1} B'C' = CD^{-1} C'D' = \dots = YZ^{-1} Y'Z' = ZA^{-1} Z'A'$$

Here also, the modulus of r is the size ratio of both polygons and the argument is the angle of rotation. The fact that the homologous exterior and interior angles are equal for directly similar polygons is trivial because:

$$B'A' B'C'^{-1} = BA BC^{-1} \quad \Rightarrow \quad \text{angle } A'B'C' = \text{angle } ABC$$

$$C'B' C'D'^{-1} = CB CD^{-1} \quad \Rightarrow \quad \text{angle } B'C'D' = \text{angle } BCD \quad \text{etc.}$$

⁴ A direct similarity is also called a *similitude* and an opposite similarity sometimes an *antisimilitude*.

The direct similarity is an equivalence relation since it has the reflexive, symmetric and transitive properties. This means that there are classes of equivalence with directly similar figures.

A similitude with $|r|=1$ is a *displacement*, since both polygons have the same size and orientation.

Two triangles ABC and $A'B'C'$ are *oppositely similar* and the sides denoted with the same letters are *homologous* if:

$$AB BC^{-1} = (A'B' B'C'^{-1})^* = B'C'^{-1} A'B'$$

where the asterisk denotes the hyperbolic conjugate. The former equality cannot be arranged into quotients of pairs of homologous sides as done before. Because of this, the similarity ratio cannot be defined for the opposite similarity but only the size ratio, which is the quotient of the lengths of any two homologous sides. An opposite similarity is always the composition of a reflection in any line and a direct similarity.

$$AB BC^{-1} = v^{-1} A'B' B'C'^{-1} v \Leftrightarrow BC^{-1} v^{-1} B'C' = AB^{-1} v^{-1} A'B' \Leftrightarrow$$

$$BC^{-1} (v^{-1} B'C' v) = AB^{-1} (v^{-1} A'B'^{-1} v) = r$$

where r is the ratio of a direct similarity whose argument is not defined but depends on the direction vector v of the reflection axis. Notwithstanding, this expression allows to define the opposite similarity of two polygons. So two polygons $ABC...Z$ and $A'B'C'...Z'$ are oppositely similar and the sides denoted with the same letters are homologous if for any hyperbolic vector v the following equalities are fulfilled:

$$AB^{-1} (v^{-1} A'B'^{-1} v) = BC^{-1} (v^{-1} B'C' v) = \dots = ZA^{-1} (v^{-1} Z'A' v)$$

that is, if after a reflection one polygon is directly similar to the other. The opposite similarity is not reflexive nor transitive and there are not classes of oppositely similar figures.

An opposite similarity with $|r|=1$ is called a *reversal*, since both polygons have the same size but opposite orientations.

Obviously the plot of similar figures on the hyperbolic plane breaks our Euclidean intuition about figures with the same form. The algebraic definition is very rigorous and clear, but we must change our visual illusions. So I recommend the exercise 13.6.

This chapter is necessarily unfinished because the geometry of the hyperbolic plane can be developed and studied with the same extension and profundity as for the Euclidean plane. Then, it is obvious that many theorems should follow in the same way as Euclidean geometry. As a last example I will define the power of a point with respect to a hyperbola.

Power of a point with respect to a hyperbola with constant radius

The locus of the points placed at a fixed distance r from a given point O is an equilateral hyperbola centred at this point with equation:

$$|OP| = r \quad (x - x_o)^2 - (y - y_o)^2 = r^2$$

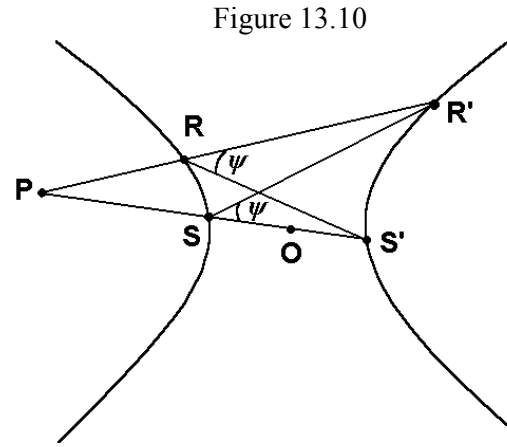
The *power of a point* with respect to this hyperbola is the product of both oriented distances from P to the intersections R and R' of a line passing through P with the hyperbola. The power of a point is constant for every line of the pencil of lines of P . In the proof the inscribed angle theorem is used (figure 13.10): the angles $S'RR'$ and $S'SR'$ are equal so the triangles SPR' and RPS' are oppositely similar. Then:

$$PR' PS^{-1} = PR^{-1} PS' \Rightarrow PR PR' = PS' PS$$

Developing the product of distances on the line passing through the centre of the hyperbola we find:

$$PS PS' = (PO + OS)(PO + OS') = PO^2 + OS OS' = (x_p - x_o)^2 - (y_p - y_o)^2 - r^2$$

that is, the power of a point is obtained by substitution of the coordinates of P on the hyperbola equation.



Exercises

13.1 Let $A = (2, 2)$, $B = (1, 0)$ and $C = (5, 3)$ be the vertices of a hyperbolic triangle. Calculate all the sides and angles and also the area.

13.2 Turn the vector $2 e_2 + e_{21}$ through an angle $\psi = \log 2$. Do a reflection in the direction $3 e_2 - e_{21}$. Make also an inversion with radius 3.

13.3 Find the direction and normal vectors of the line $y = 2x + 1$ and calculate the perpendicular line passing through the point $(3, 1)$.

13.4 Calculate the power of the point $P = (-7, 3)$ with respect to the hyperbola $x^2 - y^2 = 16$ using the intersections of the lines $y = 3$, $y = -3x - 18$ and that passing through the centre of the hyperbola $y = -3x / 7$. See that in all cases the power of P is equal to the value found by substitution in the Cartesian equation.

13.5 In this chapter I have deduced the law of sines and cosines. Therefore a law of tangents should be expected. Find and prove it.

13.6 Check that the triangle $A = (0, 0)$, $B = (5, 0)$ and $C = (5, 3)$ is directly similar to the triangle $A' = (0, 0)$, $B' = (25, -15)$, $C' = (16, 0)$. Find the similarity ratio and the rotation and dilation of the corresponding homothety. Draw the triangles and you will astonish.

FOURTH PART: PLANE PROJECTIONS OF TRIDIMENSIONAL SPACES

The complete study of the geometric algebra of the tridimensional spaces falls out the scope of this book. However, due to the importance of the Earth charts and of the Lobachevsky's geometry, the first one being more practical and the second one more theoretical, I have written this last section. In order to make the explanations clearer, the tridimensional geometric algebra has been reduced to the minimal concepts, enhancing the plane projections.

The geometric quality of being Euclidean or pseudo-Euclidean is not the signature $+$ or $-$ of a coordinate, but the fact that two coordinates have the same or different signature, in other words, it is a characteristic of a plane. For instance, a plane with signatures $++$ is equivalent, from a geometric point of view, to another with $--$. Therefore, only two kinds of three-dimensional spaces exist: the room space where all the planes are Euclidean (signatures $+++$ or $---$), and the pseudo-Euclidean space, which has one Euclidean plane and two orthogonal pseudo-Euclidean planes (signatures $+-$ or $+ - -$).

14. SPHERICAL GEOMETRY IN THE EUCLIDEAN SPACE

The geometric algebra of the Euclidean space

A *vector* of the Euclidean space is an oriented segment in this space with direction and sense, although it can represent other physical magnitudes such as forces, velocities, etc. The set of all the segments (geometric vectors) together with their addition (parallelogram rule) and the product by real numbers (dilation of vectors) has a structure of vector space, symbolised with V_3 . Every vector in V_3 is of the form:

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3$$

where e_i are three unitary perpendicular vectors, which form the base of the space. If we define an associative product (*geometric* or *Clifford product*) being a generalisation of that defined for the plane in the first chapter of this book, we will arrive to:

$$e_i^2 = 1 \quad \text{and} \quad e_i e_j = -e_j e_i \quad \text{for } i \neq j$$

In general, the square of a vector is the square of its modulus and perpendicular vectors anticommute whereas proportional vectors commute.

From the base vectors one deduces that the geometric algebra generated by the space V_3 has eight components:

$$Cl(V_3) = Cl_{3,0} = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} \rangle$$

Let us see with more detail the product of two vectors:

$$\begin{aligned} v w = (v_1 e_1 + v_2 e_2 + v_3 e_3)(w_1 e_1 + w_2 e_2 + w_3 e_3) = & v_1 w_1 + v_2 w_2 + v_3 w_3 \\ & + (v_2 w_3 - v_3 w_2) e_{23} + (v_3 w_1 - v_1 w_3) e_{31} + (v_1 w_2 - v_2 w_1) e_{12} \end{aligned}$$

The product (or quotient) of two vectors is said a *quaternion*¹. The quaternions are the even subalgebra of $Cl_{3,0}$ that generalise the complex numbers to the space. Splitting a quaternion in the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$v \wedge w = (v_2 w_3 - v_3 w_2) e_{23} + (v_3 w_1 - v_1 w_3) e_{31} + (v_1 w_2 - v_2 w_1) e_{12}$$

The *bivectors* are oriented plane surfaces and indicate the direction of planes in the space. Who be acquainted with the vector analysis will say that both vectors and bivectors are the same thing. This confusion was originated by Hamilton² himself, and continued by the founders of vector analysis, Gibbs and Heaviside. However, vectors and bivectors are different things just as physicists have experienced and know long time ago. The proper vectors are called usually “polar vectors” while the pseudo-vectors that actually are bivectors are usually called “axial vectors”. The following magnitudes are vectors: of course a geometric segment, but also a velocity, an electric field, the momentum, etc. On the other hand, the oriented area is, of course, a bivector, but also the angular momentum, the angular velocity and the magnetic field. As a criterion to distinguish both kind of magnitudes one uses the reversal of coordinates, which changes the sense of vectors while leaves bivectors invariant.

The product of two bivectors yields a real number plus a bivector. Both parts can be separated as the symmetric and antisymmetric product. The symmetric product is a real number and its negative value will be denoted here with the symbol $\bullet$ while the antisymmetric product is also a bivector and will be noted here with the symbol $\times$:

$$v \bullet w = -\frac{1}{2}(v w + w v) = v_{23} w_{23} + v_{31} w_{31} + v_{12} w_{12}$$

$$\begin{aligned} v \times w &= -\frac{1}{2}(v w - w v) \\ &= (v_{31} w_{12} - v_{12} w_{31}) e_{23} + (v_{12} w_{23} - v_{23} w_{12}) e_{31} + (v_{23} w_{31} - v_{31} w_{23}) e_{12} \end{aligned}$$

$$v w = -v \bullet w - v \times w$$

Let us see what happens with the outer product of three vectors. According to the extension theory of Grassmann, the product $u \wedge v \wedge w$ is the oriented volume generated by the surface represented by the bivector $u \wedge v$ when it is translated parallelly along the segment w :

¹ Hamilton discovered the quaternions in October 16th 1843 and defined them as quotients of two vectors. From this definition he deduced the properties of the product of quaternions. I recommend you the reading of the initial chapters of the *Elements of Quaternions* because of its pedagogic importance.

² This confusion is due to the fact that vectors and bivectors are dual spaces of the algebra $Cl_{3,0}$. However, this duality does not exist at higher dimensions, although there is also duality among other spaces.

$$u \wedge v \wedge w = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} e_{123}$$

Finally, let us see how is the product of three vectors u , v and w . The vector v can be resolved into a component coplanar with u and w and another component perpendicular to the plane u - w :

$$u \vee w = u \vee_{\parallel} w + u \vee_{\perp} w$$

Now let us analyse the permutative property. In the plane we found $u \vee w - w \vee u = 0$. In the space the permutative property becomes³:

$$\begin{aligned} u \vee w - w \vee u &= u \vee_{\perp} w - w \vee_{\perp} u = v_{\perp} (-u \wedge w + w \wedge u) \\ &= -2 v_{\perp} u \wedge w = -2 v \wedge u \wedge w = 2 u \wedge v \wedge w \end{aligned}$$

I take the same algebraic hierarchies as in chapter 1: all the abridged products must be operated before the geometric product, convention which is coherent with the fact that in many algebraic situations, the abridged products must be developed in sums of geometric products.

Spherical trigonometry

In this section the relations for the sides and angles of the spherical triangles are deduced. I will take for convenience the sphere having unity radius, although the trigonometric identities are equally valid for a sphere of any radius.

Let us consider any three points A , B and C on the sphere with unity radius centred at the origin (figure 14.1). Then $|A| = |B| = |C| = 1$. The angles formed by each pair of sides will be denoted by α , β and γ , and the sides respectively opposite to these angles will be symbolised by a , b and c respectively. Then a is the arc of the great circle passing through the points B and C , that is, the angle between these vectors:

$$\sin a = |B \wedge C|$$

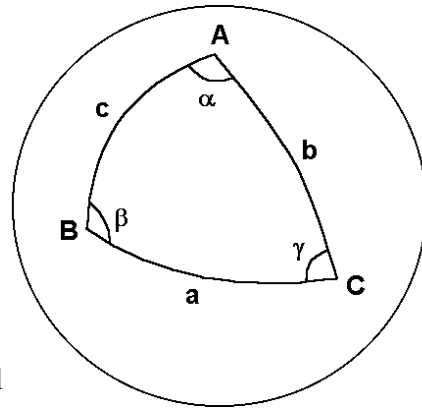


Figure 14.1

³ From this result it follows that $u \wedge v \wedge w = (u \vee w - u \vee w + v \vee u - v \vee u + w \vee u - w \vee u) / 6$. In other words, the outer product is a fully antisymmetric product. However although being beautiful, it is not so useful for geometric algebra as the permutative property $u \wedge v \wedge w = (u \vee w - w \vee u)/2$.

Note that in this equality the sine is positive and therefore $\alpha \leq \pi$. So, the spherical trigonometry is deduced for *strict triangles*, those having $a, b, c \leq \pi$.

Also, α is the angle between the sides b and c of the triangle, that is, the angle between the planes passing through the origin, A and B , and the origin, A and C . Since the direction of a plane is given by its bivector, which can be obtained through the outer product, we have:

$$\sin \alpha = \frac{|(A \wedge B) \times (A \wedge C)|}{|A \wedge B| |A \wedge C|}$$

Now we write the products of the numerator using the geometric product:

$$\begin{aligned} \sin \alpha &= \frac{|-(AB - BA)(AC - CA) + (AC - CA)(AB - BA)|}{8|A \wedge B| |A \wedge C|} \\ \sin \alpha &= \frac{|-ABAC + ABCA + BA^2C - BACA + ACAB - ACBA - CA^2B + CAB A|}{8|A \wedge B| |A \wedge C|} \end{aligned}$$

We extract the vector A as common factor at the left, but without writing it because $|A| = 1$:

$$\sin \alpha = \frac{|-BAC + BCA + ABC - A^{-1}BAC + CAB - CBA - ACB + A^{-1}CAB A|}{8|A \wedge B| |A \wedge C|}$$

Applying the permutative property to the suitable pairs of products, we have:

$$\sin \alpha = \frac{|6A \wedge B \wedge C + 2A^{-1}A \wedge B \wedge C A|}{8|A \wedge B| |A \wedge C|} = \frac{|A \wedge B \wedge C|}{|A \wedge B| |A \wedge C|}$$

since the volume $A \wedge B \wedge C$ is a pseudoscalar, which commutes with all the elements of the algebra. Now the *law of sines for spherical triangles* follows:

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c} = \frac{|A \wedge B \wedge C|}{|A \wedge B| |B \wedge C| |C \wedge A|} \quad (I)$$

Let us see the law of cosines. Since $e_{23}^2 = e_{31}^2 = e_{12}^2 = -1$ then:

$$\cos \alpha = \frac{(A \wedge B) \bullet (A \wedge C)}{|A \wedge B| |A \wedge C|} = \frac{(A \wedge B) \bullet (A \wedge C)}{\sin c \sin b}$$

$$\sin b \sin c \cos \alpha = -\frac{1}{8}[(AB - BA)(AC - CA) + (AC - CA)(AB - BA)]$$

$$= -\frac{1}{8}(A B A C - A B C A - B A^2 C + B A C A + A C A B - A C B A - C A^2 B + C A B A)$$

Now taking into account that:

$$4 A \cdot B A \cdot C = (A B + B A)(A C + C A) = A B A C + A B C A + B A^2 C + B A C A$$

But also:

$$4 A \cdot B A \cdot C = (A C + C A)(A B + B A) = A C A B + A C B A + C A^2 B + C A B A$$

and adding the needed terms, we find:

$$\sin b \sin c \cos \alpha = -\frac{1}{8}(8 A \cdot B A \cdot C - 2 A B C A - 2 B A^2 C - 2 A C B A - 2 C A^2 B)$$

Extracting common factors and using $A^2 = B^2 = 1$, we may write:

$$\sin b \sin c \cos \alpha = -\frac{1}{8}(8 A \cdot B A \cdot C - 2 A (B C + C B) A - 2 (B C + C B)) = -A \cdot B A \cdot C + B \cdot C$$

$$\sin b \sin c \cos \alpha = -\cos c \cos b + \cos a$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha \quad (\text{II})$$

which is the *law of cosines for sides*. The substitution of $\cos c$ by means of the law of cosines gives:

$$\cos a = \cos b (\cos a \cos b + \sin a \sin b \cos \gamma) + \sin b \sin c \cos \alpha$$

$$\cos a (1 - \cos^2 b) = \cos b \sin a \sin b \cos \gamma + \sin b \sin c \cos \alpha$$

and the simplification of $\sin b$:

$$\cos a \sin b = \cos b \sin a \cos \gamma + \sin c \cos \alpha$$

The substitution of $\sin c = \sin a \sin \gamma / \sin \alpha$ yields:

$$\cos a \sin b = \cos b \sin a \cos \gamma + \frac{\sin a \sin \gamma \cos \alpha}{\sin \alpha}$$

Dividing by $\sin a$:

$$\cot a \sin b = \cos b \cos \gamma + \sin \gamma \cot \alpha \quad (\text{III})$$

The dual spherical triangle

The perpendicular to the plane containing each side of a spherical triangle cuts the spherical surface in a point. The three points A' , B' and C' obtained in this geometric way form the *dual triangle*. The algebraic way to calculate them is the duality operation, which maps bivectors into the perpendicular vectors. The dual of any element is obtained as the product by the pseudoscalar unity $-e_{123}$, which commutes with all the elements of the algebra:

$$A' = -e_{123} \frac{B \wedge C}{|B \wedge C|} \quad B' = -e_{123} \frac{C \wedge A}{|C \wedge A|} \quad C' = -e_{123} \frac{A \wedge B}{|A \wedge B|}$$

The inner product of two vectors yields:

$$B' \cdot C' = \frac{(C \wedge A) \bullet (A \wedge B)}{|C \wedge A| |A \wedge B|} = - \frac{(A \wedge B) \bullet (A \wedge C)}{|A \wedge B| |A \wedge C|} \Leftrightarrow \cos a' = -\cos \alpha$$

which shows that the angle a' and α are supplementary, and so also the other angles:

$$a' = \pi - \alpha \quad b' = \pi - \beta \quad c' = \pi - \gamma$$

It is trivial that the dual of the dual triangle is the first triangle, and hence:

$$\alpha' = \pi - a \quad \beta' = \pi - b \quad \gamma' = \pi - c$$

We may apply the laws of sines and cosines to the dual triangle. The law of sines is self-dual and may be written in a more symmetric form:

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c} = \frac{|A \wedge B \wedge C|}{|A' \wedge B' \wedge C'|} \quad (\text{I})$$

On the other hand, the law of cosines yields a new result when applying duality:

$$-\cos \alpha' = \cos \gamma' \cos \beta' - \sin \beta' \sin \gamma' \cos a'$$

Removing the marks, since this law must be valid for any spherical triangle, we find the *law of cosines for angles*:

$$\cos \alpha = -\cos \gamma \cos \beta + \sin \beta \sin \gamma \cos a \quad (\text{IV})$$

Also, applying the equality (III) to the dual triangle we find:

$$\cot \alpha \sin \beta = -\cos \beta \cos c + \sin c \cot a \quad (\text{V})$$

The five Bessel's equalities (I to V) allow to solve every spherical triangle knowing any three of its six elements.

Right spherical triangles and Napier's rule

For the case of a right angle spherical triangle, the five Bessel's formulas are reduced to a simpler form, and then they may be remembered with the help of the Napier's pentagon rule (figure 14.2). Draw the angles and the sides of the triangle following the order of the perimeter removing the right angle and writing instead of the legs (the sides adjacent to the right angle) the complementary arcs. Then follow the Pentagon rules:

- 1) The cosine of every element is equal to the product of the cotangents of the adjacent elements.
- 2) The cosine of every element is equal to the product of the sines of the nonadjacent elements.

For example:

$$\cos a = \sin\left(\frac{\pi}{2} - b\right) \sin\left(\frac{\pi}{2} - c\right) = \cos b \cos c$$

$$\text{also: } \cos\left(\frac{\pi}{2} - b\right) = \cot \gamma \cot\left(\frac{\pi}{2} - c\right) \quad \Rightarrow \quad \sin b = \cot \gamma \operatorname{tg} c$$

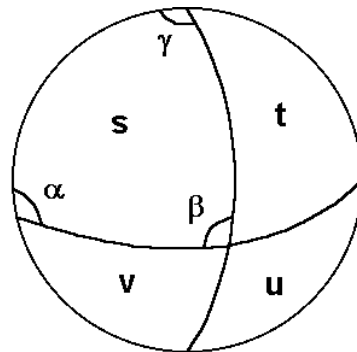
This rule is applied to the right side triangles in the same way: remove the right side and write the complementary of the adjacent angles.

Area of a spherical triangle

A *lune* is a two-sided polygon on the sphere defined by two great circles. The area of a lune is proportional to the angle α between both great circles. For an angle $\pi/2$ its area is π , therefore the area of a lune with angle α is 2α . Now let us consider in the sphere shown by figure 14.3 the three lunes having the angles α , β and γ . Then:

$$\begin{cases} s + t = 2\alpha \\ s + u' = 2\beta \\ s + v = 2\gamma \end{cases}$$

Figure 14.3



where u' is the area of the antipodal triangle of u . Both triangles u and u' have the same angles and area and the system can be rewritten:

$$\begin{cases} s + t = 2\alpha \\ s + u = 2\beta \\ s + v = 2\gamma \end{cases}$$

Adding the three equations, we find:

$$3s + t + u + v = 2(\alpha + \beta + \gamma)$$

The four triangles s , t , u and v fill an hemisphere:

$$s + t + u + v = 2\pi$$

so the area of the triangle s is the *spherical excess*, that is, the addition of the three angles minus π :

$$s = \alpha + \beta + \gamma - \pi$$

Properties of the projections of the spherical surface

No chart of the spherical surface preserving the scale of distances everywhere exists, that is, we cannot depict any map with distances proportional to those measured on the sphere. The *distortion* is the variation of the scale and the angles. Usually there is a line of zero distortion, where the scale is constant.

Since the scale of distances is never preserved for all the points, projections with other interesting properties have been searched in cartography. A projection is said to be *conformal* if it preserves in the map the angles between great circles on the sphere. A projection is said to be *equivalent* if it preserves the area, that is if the figures on the sphere are projected into figures on the map having the same area. A projection is said to be *equidistant* if the scale of distances is preserved, not everywhere but on the line perpendicular to the line of zero distortion, or radially outwards from a point of zero distortion.

The more general concept of projection is any one-to-one mapping of any point (x, y, z) on the sphere into a point (u, v) on the plane, although the main types of projections are azimuthal, cylindrical and conic. An *azimuthal* projection is a standard projection into a plane, which may be considered touching the sphere. In the tangency point there is zero distortion and the bearings or azimuths from this point are correctly shown. A *cylindrical* projection is a projection into a cylindrical surface around the sphere that will be unrolled. A *conic* projection is a projection into a conical surface tangent to the sphere that will also be unrolled.

Now, I review the main and more used projections beginning with the azimuthal projections.

The central or gnomonic projection

Let us consider the sphere with unity radius (figure 14.4) centred at the origin. Any point on the sphere has the coordinates (x, y, z) fulfilling:

$$x^2 + y^2 + z^2 = 1$$

Every point on the upper hemisphere is projected into another point on the plane $z = 1$ using as centre of projection the centre of the sphere. Let u and v be the Cartesian coordinates on the projection plane. Taking similar triangles the following relations are found:

$$u = \frac{x}{z} \quad v = \frac{y}{z}$$

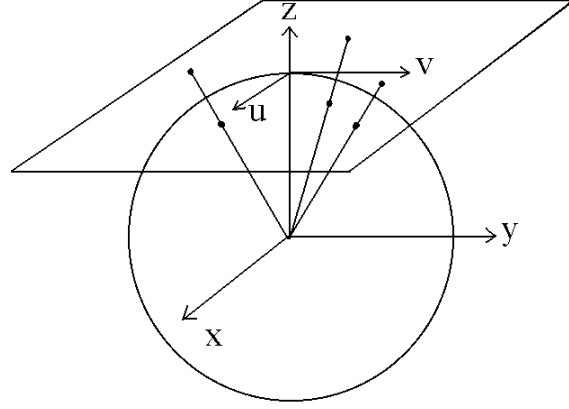


Figure 14.4

from where we obtain:

$$x = \frac{u}{\sqrt{u^2 + v^2 + 1}} \quad y = \frac{v}{\sqrt{u^2 + v^2 + 1}} \quad z = \frac{1}{\sqrt{u^2 + v^2 + 1}}$$

The differential of the arc length is obtained through the differentiation of the above relationships:

$$ds = dx e_1 + dy e_2 + dz e_3$$

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{(1 + v^2)du^2 - 2uv du dv + (1 + u^2)dv^2}{(1 + u^2 + v^2)^2}$$

The geodesics of the sphere are the great circles, which are the intersections with planes passing through its centre. These planes cut the projection plane in straight lines, which are the projections of the geodesics. In other words, any great circle is projected into a line on the projection plane. Taking as equation of the line:

$$v = k u + l$$

with k and l constant, the substitution in ds gives:

$$ds = \frac{\sqrt{1 + k^2 + l^2}}{u^2(1 + k^2) + 2kl u + 1 + l^2} du$$

By integration we arrive to the following primitive:

$$s = \arctg \frac{u(1 + k^2) + kl}{\sqrt{1 + k^2 + l^2}} + \text{const}$$

The arc length between two points on this great circle is the difference of this primitive between both points. However it is more advantageous to write it using cosines instead of tangents by means of the trigonometric identity:

$$\cos(s_B - s_A) \equiv \frac{1 + \operatorname{tg} s_A \operatorname{tg} s_B}{\sqrt{1 + \operatorname{tg}^2 s_A} \sqrt{1 + \operatorname{tg}^2 s_B}}$$

After removing k and l using the equation of the line, we arrive to:

$$\begin{aligned} \cos(s_B - s_A) &= \frac{1 + u_A u_B + v_A v_B}{\sqrt{1 + u_A^2 + v_A^2} \sqrt{1 + u_B^2 + v_B^2}} \\ &= \frac{(u_A e_1 + v_A e_2 + e_3) \cdot (u_B e_1 + v_B e_2 + e_3)}{|u_A e_1 + v_A e_2 + e_3| |u_B e_1 + v_B e_2 + e_3|} \end{aligned}$$

a trivial result because the arc length is the angle between the position vectors of both points, and also of the proportional vectors going to the projection plane. However, the interest of this result is its analogy with the result found for the hyperboloidal surface.

From this value of the cosine, we may obtain the sine of the arc:

$$\begin{aligned} \sin(s_B - s_A) &= \sqrt{1 - \cos^2(s_B - s_A)} = \frac{\sqrt{(u_A v_B - u_B v_A)^2 + (u_A - v_A)^2 + (u_B - v_B)^2}}{\sqrt{1 + u_A^2 + v_A^2} \sqrt{1 + u_B^2 + v_B^2}} \\ &= \frac{|(u_A e_1 + v_A e_2 + e_3) \wedge (u_B e_1 + v_B e_2 + e_3)|}{|u_A e_1 + v_A e_2 + e_3| |u_B e_1 + v_B e_2 + e_3|} \end{aligned}$$

which is also a trivial result, since the sine of the angle is proportional to the modulus of the outer product.

Let us see the area function. The differential of the area is easily obtained taking into account that it is a bivector and using the outer product of the differentials of the coordinates:

$$dA = \sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2} = \frac{du \wedge dv}{(1 + u^2 + v^2)^{3/2}}$$

This result shows that the central projection is not *equivalent* and the distortion increases with the distance to the origin.

Let us consider a plane passing through the centre of the sphere, which cuts its surface in the great circle determined by the equation system:

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ a x + b y + z = 0 \quad a, b \text{ real} \end{cases}$$

Then the angle between two great circles is the angle between both central planes:

$$\cos \alpha = \frac{(a e_{23} + b e_{31} + e_{12}) \bullet (a' e_{23} + b' e_{31} + e_{12})}{|a e_{23} + b e_{31} + e_{12}| |a' e_{23} + b' e_{31} + e_{12}|} = \frac{a a' + b b' + 1}{\sqrt{a^2 + b^2 + 1} \sqrt{a'^2 + b'^2 + 1}}$$

The sides of a spherical triangle are great circles; therefore the central projection of a spherical triangle is a triangle whose sides are straight lines.

Stereographic projection

In the stereographic projection the point of view is placed on the spherical surface. As before, every point (x, y, z) on the sphere with unity radius centred at the origin fulfils the equation:

$$x^2 + y^2 + z^2 = 1$$

We project the spherical surface into the plane $z = 0$ locating the centre of projection at the pole $(0, 0, -1)$ (figure 14.5). The upper hemisphere is projected inside the circle of unity radius while the lowest hemisphere is projected outside. If u and v are the Cartesian coordinates on the projection plane, we have by similar triangles:

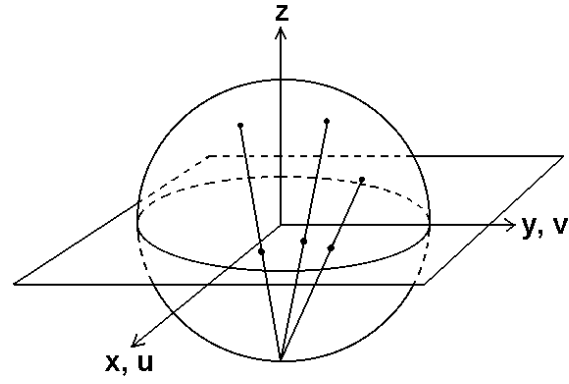


Figure 14.5

$$\frac{x}{u} = z + 1 \quad \frac{y}{v} = z + 1$$

Using the equation of the sphere one arrives to:

$$x = \frac{2u}{1 + u^2 + v^2} \quad y = \frac{2v}{1 + u^2 + v^2} \quad z = \frac{2}{1 + u^2 + v^2} - 1$$

from where the differential of the arc length is obtained:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{4(du^2 + dv^2)}{(1 + u^2 + v^2)^2}$$

Now we see that this projection is not equidistant and the distortion increases with the distance to the origin. The factor 4 indicates that the lengths at the origin of coordinates on the plane are the half of the lengths on the sphere. Taking instead of the plane $z = 0$, the plane $z = 1$ this factor becomes 1. Then, we can state correctly the scale of the chart (for example, a polar chart).

The geodesic lines (great circles) are intersection of the sphere surface with planes passing through the origin, which have the equation:

$$z = a x + b y$$

The substitution by the stereographic coordinates yields:

$$(u + a)^2 + (v + b)^2 = a^2 + b^2 + 1$$

which is the equation of a circle centred at $(-a, -b)$ with radius $r = \sqrt{a^2 + b^2 + 1}$. Observe that this radius is the hypotenuse of a right angle triangle having as legs the distance to the origin and the unity. That is, the great circles on the sphere are shown in the stereographic projection as circles that intersect the circle $x^2 + y^2 = 1$ in extremes of diameters (figure 14.6).

Usually only the projection of the upper hemisphere is used, so that the great circles are represented as circle arcs inside the circle $x^2 + y^2 \leq 1$. The angle α between two of these circles, as explained in the page 89, is obtained from their radii r and r' and centres O and O' through:

$$\cos \alpha = \frac{r^2 + r'^2 - (O - O')^2}{2 r r'} = \frac{a a' + b b' + 1}{\sqrt{a^2 + b^2 + 1} \sqrt{a'^2 + b'^2 + 1}}$$

Just this is the angle between the planes $a x + b y - z = 0$ and $a' x + b' y - z = 0$, that is, the angle between the two great circles represented by the projected circles. Therefore, the stereographic projection is a conformal projection of the spherical surface.

If we calculate the differential of area we find:

$$dA = \frac{4 du \wedge dv}{(1 + u^2 + v^2)^2}$$

Now we see that this projection is not equivalent since distortion of areas increases with the distance from the origin (as commented above, the factor 4 becomes 1 projecting into the plane $z=1$).

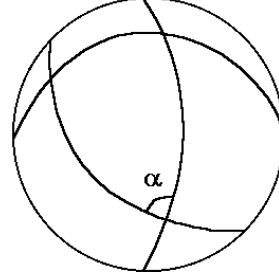
Orthographic projection

The orthographic projection of the sphere is a parallel projection (the point of view is placed at the infinity). If we make a projection parallel to the z -axis upon the plane $z = 0$, the Cartesian coordinates on the map are identical to x, y and we have:

$$z = \sqrt{1 - x^2 - y^2} \quad dz = -\frac{x dx + y dy}{\sqrt{1 - x^2 - y^2}}$$

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{dx^2 + dy^2 + 2 x y dx dy}{1 - x^2 - y^2}$$

Figure 14.6



$$dA = \sqrt{\frac{1+x^2+y^2}{1-x^2-y^2}} dx \wedge dy$$

Introducing the distance r to the origin of coordinates and the angle φ with respect to the x -axis, we have:

$$x = r \cos \varphi \quad y = r \sin \varphi$$

$$ds^2 = \frac{dr^2 + r^2 d\varphi^2 + 2r dr d\varphi}{1-r^2}$$

An example of orthographic projection is the Earth image that appears in many TV news.

Spherical coordinates and cylindrical equidistant (Plate Carrée) projection

For a sphere with unity radius, the spherical coordinates⁴ are related with the Cartesian coordinates by means of:

$$x = \sin \theta \cos \varphi$$

$$y = \sin \theta \sin \varphi$$

$$z = \cos \theta$$

Then the differential of arc length is:

$$ds = (\cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi) e_1 + (\cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi) e_2 - \sin \theta d\theta e_3$$

Introducing the unitary vectors e_θ and e_φ as:

$$e_\theta = \cos \theta \cos \varphi e_1 + \cos \theta \sin \varphi e_2 - \sin \theta e_3$$

$$e_\varphi = -\sin \varphi e_1 + \cos \varphi e_2$$

the differential of arc length in spherical coordinates becomes:

$$ds = d\theta e_\theta + \sin \theta d\varphi e_\varphi$$

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$$

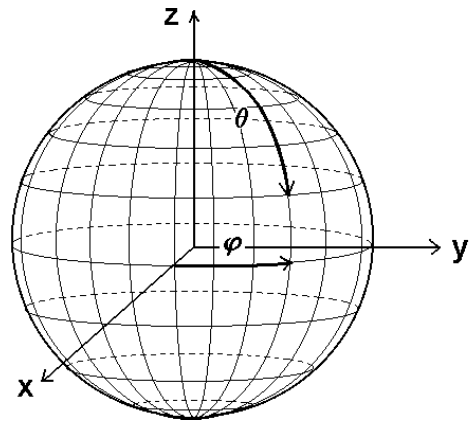


Figure 14.7

⁴ For geographical coordinates, θ is the colatitude and φ the longitude.

Note that e_θ and e_φ are orthogonal vectors, since their inner product is zero. At $\theta = 0$, ds depends only on $d\theta$ and there is a pole. Then θ is the arc length from the pole to the given point (figure 14.7), while φ is the arc length over the equator ($\theta = \pi/2$, $\sin\theta = 1$). The meridians ($\varphi = \text{constant}$) are geodesics, but the parallels ($\theta = \text{constant}$) are not. Exceptionally, the equator is also a geodesic.

In the Plate Carrée projection $u = \varphi$ and $v = \pi/2 - \theta$ (latitude) so that the meridians and parallels are shown in a squared graticule. This projection is equidistant for any meridian whereas the distortion in the parallels increases, as well as for every cylindrical projection, as we separate from the equator. Let us see the area:

$$dA = \sin \theta \, d\theta \wedge d\varphi = \cos v \, du \wedge dv$$

The ratio of the real area with respect to the represented area on the chart is equal to the cosine of the latitude and therefore the projection is not equivalent.

Mercator's projection

The Mercator's projection⁵ is defined as the cylindrical conformal projection. If we wish to preserve the angles between curves, we must enlarge the meridians by the same amount as the parallels are enlarged in a cylindrical projection, that is, by the factor $1/\sin \theta$ (the secant of the geographical latitude):

$$dv = \frac{-d\theta}{\sin \theta} \quad \Rightarrow \quad v = -\log \operatorname{tg} \frac{\theta}{2}$$

The differential of arc length is:

$$ds^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 = \sin^2 \theta (du^2 + dv^2) = \frac{4 \exp(2v)}{(\exp(2v) + 1)^2} (du^2 + dv^2)$$

$$ds = \frac{2 \exp(v)}{\exp(2v) + 1} \sqrt{du^2 + dv^2}$$

where the distances are increased in an amount independent of the direction and proportional to the inverse of the sine of θ . The differential of area is:

$$dA = \sin \theta \, d\theta \wedge d\varphi = \frac{4 \exp(2v)}{(\exp(2v) + 1)^2} du \wedge dv$$

⁵ The difference between the U.T.M. (Universal Transverse Mercator) projection and the Mercator planisphere is not geometric but geographic: in the U.T.M. the cylinder of projection is tangent to a meridian instead of the equator. All the Earth has been divided in zones of 6° of longitude, where the cylinder of projection is tangent to the central meridian (3°, 9°, 15° ...).

Peters' projection

The Peters' projection is the cylindrical equivalent projection. If we wish to preserve area, we must shorten the meridians in the same amount as the parallels are enlarged in the cylindrical projection, what yields:

$$dv = -\sin \theta d\theta \quad \Rightarrow \quad v = \cos \theta$$

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 = (1 - v^2) du^2 + \frac{dv^2}{1 - v^2}$$

$$dA = \sin \theta d\theta \wedge d\varphi = du \wedge dv$$

which displays clearly the equivalence of the projection. Observe that $v = \cos \theta$ means the sphere is projected following planes perpendicular to the cylinder of projection (figure 14.8).

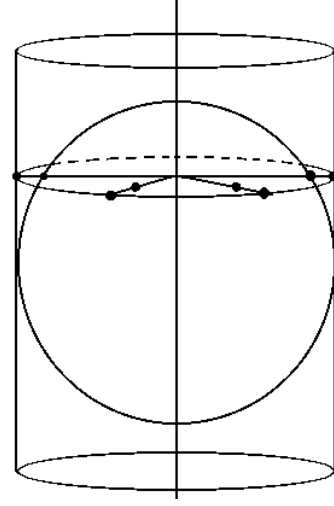


Figure 14.8

Conic projections

These projections are made into a cone surface tangent to the sphere (figure 14.9). Because the cone surface unrolled is a plane circular sector, they are often used to display middle latitudes, while the azimuthal projections are mainly used for poles. In a conic projection a small circle (a parallel) is shown as a circle with zero distortion. The characteristic parameter of a conic projection is the constant of the cone $n = \cos \theta_0$, being θ_0 the angle of inclination of the generatrix of the cone and also the angle from the axis of the cone to any point of tangency with the sphere. Since the graticule of the conic projections is radial, to use the radius r and the angle χ is more convenient:

$$dr = f(\theta) d\theta \quad d\chi = n d\varphi$$

The differentials of arc length and area for a conic projection are:

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 = \frac{dr^2}{[f(\theta)]^2} + \frac{\sin^2 \theta}{n^2} d\chi^2$$

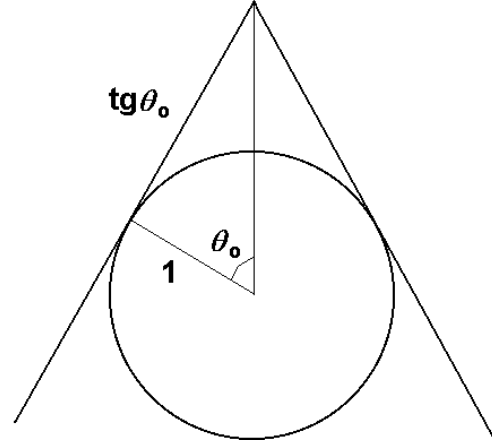
$$dA = \sin \theta d\theta \wedge d\varphi = \frac{\sin \theta}{n f(\theta)} dr \wedge d\chi$$

Let us see as before the three special cases: equidistant, conformal and equivalent projections. The differential of area for polar coordinates r, χ is $dA = r dr \wedge d\chi$. If the projection is equivalent, we must identify both dA to find :

$$\frac{d}{d\theta} \left[\frac{\sin \theta}{n f(\theta)} \right] = f(\theta) \quad \Rightarrow \quad f(\theta) = \frac{1}{\sqrt{n}} \cos \frac{\theta}{2} \quad \text{and} \quad r = \frac{2}{\sqrt{n}} \sin \frac{\theta}{2}$$

$$ds^2 = \frac{n}{1 - \frac{n r^2}{4}} dr^2 + \frac{1 - \frac{r^2 n}{4}}{n} r^2 d\chi^2$$

Figure 14.9



If the projection is equidistant, the meridians have zero distortion so $d\theta = dr/n$ and:

$$ds^2 = \frac{1}{n^2} \left(dr^2 + \sin^2 \left(\frac{r}{n} \right) d\chi^2 \right)$$

If the projection is conformal then $ds^2 \propto dr^2 + r^2 d\chi^2$ so:

$$ds^2 = \frac{\sin^2 \theta}{n^2 r^2} (dr^2 + r^2 d\chi^2)$$

Solving the differential equation:

$$d\theta = \frac{\sin \theta}{n r} dr$$

with the boundary condition $\text{tg } \theta_0 = r_0$ as shown by the figure 14.9 we find the Lambert's conformal projection:

$$r = \text{tg } \theta_0 \left(\frac{\text{tg } \frac{\theta}{2}}{\text{tg } \frac{\theta_0}{2}} \right)^n$$

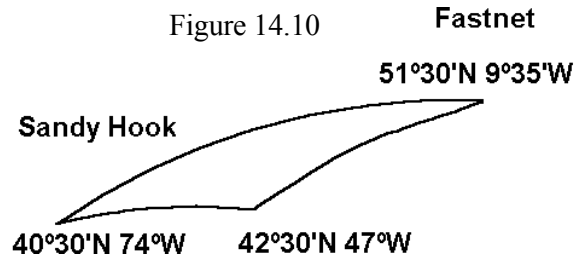
Exercises

14.1 We have seen the gnomonic projection of great circles being always straight lines. Then, why does the shadow of a gnomon of a sundial follow a hyperbola on a plane surface instead of a line during a day? Why does this hyperbola become a straight line the March 21 and September 23?

14.2 Three points on the sphere are projected by means of the stereographic projection into a circle of unity radius with the coordinates $A = (-0.5, 0.5)$, $B = (0, -2/3)$ and $C = (2/3, 0)$. Calculate the sides, angles and area of the triangle which they form.

14.3 Built at Belfast, the Titanic begun its first and last travel in Southampton the April 10th 1912. After visiting Cherbourg the Titanic weighed anchor in the Cork harbour

with bearing New York the 11th April. Calculate the length of the shortest trajectory from Fastnet (Ireland) at 51°30'N 9°35'W to Sandy Hook (New York) at 40°30'N 74°W. From January 15th to July 15th the ships had to follow the orthodrome (great circle) from Fastnet to the point 42°30'N 47°W and from this point to Sandy Hook passing at twenty miles from the floating lighthouse of Nantucket. Calculate the length of the route the Titanic should have followed. Take as an averaged radius of the Earth 6366 km. The Titanic sank the 15th April at the position 41°46'N 50°14'W. Is this point on the obliged track?



14.4 The figure 14.11 is a photograph of the Hale-Bopp comet. The orientation of the camera is unknown, but the W of Cassiopeia constellation appears in the photograph. The declination D of a star is the angular distance from the celestial equator to the star (measured on the great circle passing through the celestial pole and the star). The right ascension A is the arc of celestial equator measured eastward from the vernal equinox (one of the intersections of the celestial equator with the ecliptic, also called Aries point) to the foot of the great circle passing through the star and the pole. The right ascension and declination of the stars are constant while those of the comets, planets, the Sun and the Moon are variable. The data of Cassiopeia constellation are:

star	Magnitude	A	D
α Cassiopeia (Schedir)	2.2	0h 40min 6.4s = 10.0267°	56° 29' 57" N
β Cassiopeia (Caph)	2.3	0h 8min 48.1s = 2.2004°	59° 6' 40" N
γ Cassiopeia (Tsih)	2.4	0h 56min 16.9s = 14.0704°	60° 40' 44" N
δ Cassiopeia (Ruchbah)	2.7	1h 25min 21.2s = 21.3383°	60° 11' 57" N

In your system of coordinates, take the x -axis as the Aries point and the z -axis as the north pole. Then $D = \pi/2 - \theta$ and $A = \varphi$.

- Calculate the focal distance of the photograph, that is, the distance from the point of view to the plane of the photograph. Knowing that the negative was universal (24×36 mm), calculate the focal distance of the camera.
- Calculate the orientation (right ascension and declination) of the photographic camera.
- Calculate the coordinates of the Hale-Bopp comet.

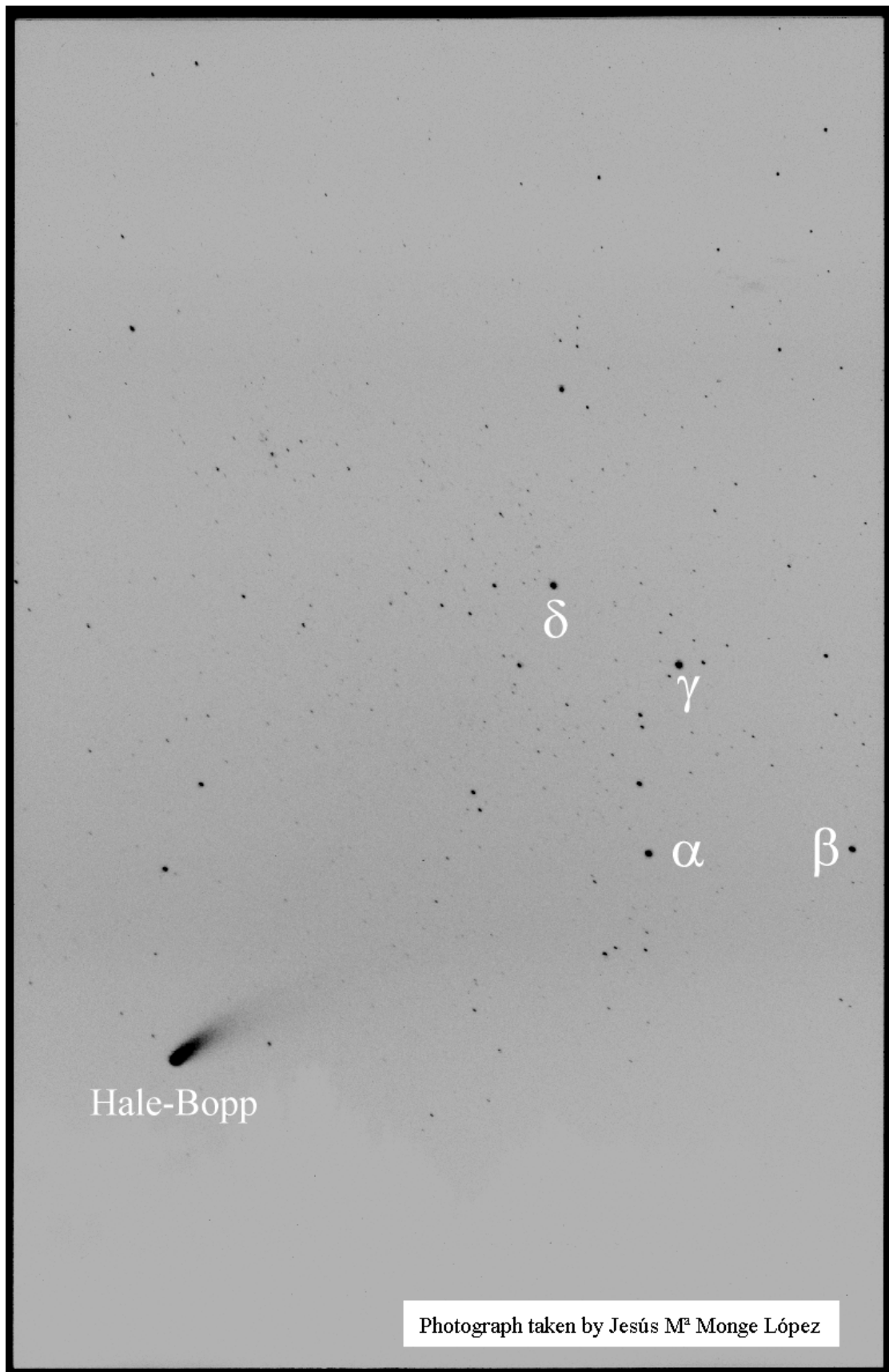


Figure 14.11

15. HYPERBOLOIDAL GEOMETRY IN THE PSEUDO-EUCLIDEAN SPACE (LOBACHEVSKY'S GEOMETRY)

The geometric algebra of the pseudo-Euclidean space

A *vector* of the pseudo-Euclidean space is an oriented segment in this space with direction and sense. The set of all segments (vectors) together with their addition (parallelogram rule) and the product by real numbers (dilation of vectors) has a structure of vector space, symbolised here with W_3 . Every vector in W_3 is of the form:

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3$$

where e_i are three unitary perpendicular vectors, which constitute the base of the space. The modulus of a vector is:

$$|v|^2 = -v_1^2 - v_2^2 + v_3^2$$

It determines the geometric properties of the space, very different from the Euclidean space. Now we define an associative product (*geometric* or *Clifford product*) being a generalisation of those defined for the Euclidean and hyperbolic planes. Imposing the condition that the square of the modulus be equal to the square of the vector, we find:

$$|v|^2 = v^2$$

$$e_1^2 = -1 \quad e_2^2 = -1 \quad e_3^2 = 1 \quad \text{and} \quad e_i e_j = -e_j e_i \text{ for } i \neq j$$

From the base vectors one deduces that the geometric algebra generated by the space W_3 has eight components:

$$Cl(W_3) = Cl_{1,2} = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} \rangle$$

Let us see with more detail the product of two vectors:

$$\begin{aligned} v w = (v_1 e_1 + v_2 e_2 + v_3 e_3)(w_1 e_1 + w_2 e_2 + w_3 e_3) = & -v_1 w_1 - v_2 w_2 + v_3 w_3 \\ & + (v_2 w_3 - v_3 w_2) e_{23} + (v_3 w_1 - v_1 w_3) e_{31} + (v_1 w_2 - v_2 w_1) e_{12} \end{aligned}$$

I shall call the product (or quotient) of two vectors a *tetranion*. The tetranions are the even subalgebra of $Cl_{1,2}$ that generalises the complex and hyperbolic numbers to the pseudo-Euclidean space. Splitting a tetranion in the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = -v_1 w_1 - v_2 w_2 + v_3 w_3$$

$$v \wedge w = (v_2 w_3 - v_3 w_2) e_{23} + (v_3 w_1 - v_1 w_3) e_{31} + (v_1 w_2 - v_2 w_1) e_{12}$$