Here also, the bivectors are oriented plane surfaces indicating the direction of planes in the pseudo-Euclidean space. As before, vectors and bivectors are different things. This fact has been experienced also by physicists: in the Minkowski's space, the electromagnetic field is a bivector while the tetrapotential is a vector. On the other hand, the oriented area is, of course, a bivector. As a criterion to distinguish both kind of magnitudes one also uses the reversal of coordinates, which changes the sense of vectors while leaves bivectors invariant.

Two vectors are said to be *orthogonal* if their inner product vanishes:

$$v \perp w \quad \Leftrightarrow \quad v \cdot w = 0$$

So the outer product is the product by the orthogonal component and the inner product is the product by the proportional component:

$$v \cdot w = v\, w_{\parallel} \qquad\qquad v \wedge w = v\, w_{\perp}$$

The product of two bivectors yields a tetranion. The real and bivector parts can be separated as the symmetric and antisymmetric product. Since:

$$e_{23}^2 = e_{31}^2 = 1 \qquad \text{and} \qquad e_{12}^2 = -1$$

the symmetric product is a real number whose negative value will be denoted here with the symbol •, whereas the antisymmetric product, denoted here with the symbol ×, is also a bivector:

$$v \bullet w = -\frac{1}{2}(v\, w + w\, v) = -v_{23}\, w_{23} - v_{31}\, w_{31} + v_{12}\, w_{12}$$

$$v \times w = -\frac{1}{2}(v\, w - w\, v)$$

$$= -(v_{31}\, w_{12} - v_{12}\, w_{31})\, e_{23} - (v_{12}\, w_{23} - v_{23}\, w_{12})\, e_{31} + (v_{23}\, w_{31} - v_{31}\, w_{23})\, e_{12}$$

$$v\, w = -v \bullet w - v \times w$$

The outer product of three vectors has the same expression as for Euclidean geometry and this is a natural outcome of the extension theory: the product $u \wedge v \wedge w$ is the oriented volume generated by the surface represented by the bivector $u \wedge v$ when it is translated parallelly along the segment w:

$$u \wedge v \wedge w = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} e_{123}$$

Finally, let us see how the product of three vectors u, v and w is. The vector v can be resolved into a component coplanar with u and v and another component perpendicular to the plane u-v:

$$u\, v\, w = u\, v_{\parallel}\, w + u\, v_{\perp}\, w$$

Now let us analyse the permutative property. In both Euclidean and hyperbolic planes we found $u\ v\ w - w\ v\ u = 0$. In the pseudo-Euclidean space the permutative property becomes:

$$u\, v\, w - w\, v\, u = u\, v_{\perp}\, w - w\, v_{\perp}\, u = v_{\perp}\left(-u\, w + w\, u\right)$$

$$= -2\, v_{\perp}\, u \wedge w = -2\, v \wedge u \wedge w = 2\, u \wedge v \wedge w$$

I take the same algebraic hierarchies as in the former chapter: all the abridged products must be operated before the geometric product, convention adequate to the fact that in many algebraic situations, the abridged products must be developed in sums of geometric products.

The hyperboloid of two sheets

According to Hilbert (*Grundlagen der Geometrie*, Anhang V) the complete Lobachevsky's "plane" cannot be represented by a smooth surface with constant curvature as proposed by Beltrami. But this result only concerns surfaces in the Euclidean space. The surface whose points are placed at a fixed distance from the origin in a pseudo-Euclidean space (the two-sheeted hyperboloid) is the surface searched by Hilbert which realises the Lobachevsky's geometry[1]. It is known that it has a characteristic distance like the radius of the sphere. Since all the spheres are similar, we need only study the unitary sphere. Likewise, all the hyperboloidal surfaces $z^2-x^2-y^2 = r^2$ are similar and the hyperboloid with unity radius (figure 15.1):

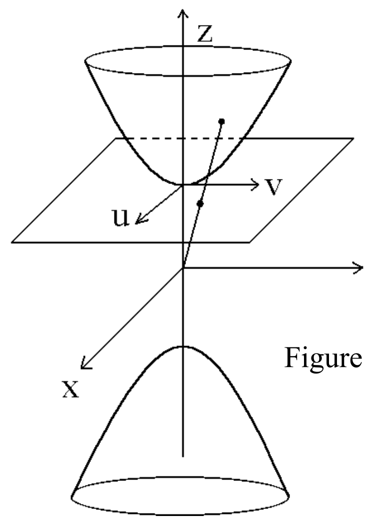


Figure 15.1

$$z^2 - x^2 - y^2 = 1$$

where (x, y, z) are Cartesian although not Euclidean[2] coordinates, suffices to study the whole Lobachesky's geometry.

[1] The reader will find a complete study of the hyperboloidal surface in Faber, *Foundations of Euclidean and Non-Euclidean Geometry*, chap. VII. "The Weierstrass model".

[2] For me, Cartesian coordinates are orthogonal coordinates in a flat space (with null curvature), independently of the Euclidean or pseudo-Euclidean nature of the space. Lines and planes (linear relations between Cartesian coordinates) can always be drawn in both spaces, although the pseudo-Euclidean orthogonality is not shown as a right angle, and the pseudo-Euclidean distance is not the viewed length. The incompleteness of Cartesian coordinates was already criticised by Leibniz in a letter to Huygens in 1679 proposing a new geometric calculus: «Car premierement je puis exprimer parfaitement par ce calcul toute la nature ou définition de la figure (ce que l'algèbre ne fait jamais, car disant que $x^2 + y^2$ *aeq.* a^2 est l'equation du cercle, il

The fact that the hyperboloidal surface owns the Lobachevsky's geometry will be evident through the different projections here reviewed. If we followed the same order as in the former chapter, we should firstly deal with the Lobachevskian trigonometry. However, the concept of arc length on the hyperboloid is much less intuitive than on the sphere. Therefore we must see the concept and determination of the arc length before studying the hyperboloidal trigonometry.

The central projection (Beltrami disk)

In this projection the centre of the hyperboloid is the centre of projection. Thus, we project every point on the upper sheet of the hyperboloid $z^2-x^2-y^2 = 1$ into another point on the plane $z = 1$ taking the origin $x = 0$, $y = 0$, $z = 0$ as centre of projection (figure 15.1). Let u and v be Cartesian coordinates on the plane $z = 1$, which touches the hyperboloid in the vertex. By similar triangles[3]:

$$u = \frac{x}{z} \qquad v = \frac{y}{z}$$

From where we obtain:

$$x = \frac{u}{\sqrt{1-u^2-v^2}} \qquad y = \frac{v}{\sqrt{1-u^2-v^2}} \qquad z = \frac{1}{\sqrt{1-u^2-v^2}}$$

The differential of the arc length upon the hyperboloid is:

$$ds = dx\, e_1 + dy\, e_2 + dz\, e_3$$

$$ds^2 = -dx^2 - dy^2 + dz^2 = -\frac{\left(1-v^2\right)du^2 + 2\,u\,v\,du\,dv + \left(1-u^2\right)dv^2}{\left(1-u^2-v^2\right)^2}$$

which is the negative value of the usual metric of the Lobachevsky's surface in the Beltrami disk. The negative sign indicates that the arc length is comparable with lengths in the x-y plane. The whole upper sheet of the hyperboloid is projected inside the Beltrami's circle $u^2 + v^2 \le 1$ with unity radius. However, if we project the one-sheeted hyperboloid $z^2 - x^2 - y^2 = -1$ into the plane $z = 1$ we find the same metric but positive, what indicates the lengths being comparable to those measured on the z-axis. All the upper half of the one-sheeted hyperboloid is projected outside the circle of unity radius, giving rise to a new geometry not studied up till now.

---

faut expliquer par la figure ce que c'est *x* et *y*.» (Josep Manel Parra, "Geometric Algebra versus numerical cartesianism" in F. Brackx, R. Delanghe, H. Serras, *Clifford Algebras and Their Applications in Mathematical Physics*).

[3] Now to consider that the legs of both right angle triangles are horizontal and vertical, and therefore perpendicular is enough for the deduction. Notwithstanding, in a pseudo-Euclidean space one must use the algebraic definition of similitude: a triangle with sides a and b is directly similar to a triangle with sides c and d if and only if $a\,c^{-1} = b\,d^{-1}$. This condition implies that both quotients of the modulus are equal and both arguments also.

Let us consider a plane passing through the origin of coordinates (central plane). Its intersection with the two-sheeted hyperboloid is the hyperbola determined by the equations system:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ z = a\,x + b\,y \qquad a, b \ \text{ real} \end{cases}$$

We search the Frenet's trihedron. By differentiation we find a linear differential equations system :

$$\begin{cases} z\,dz = x\,dx + y\,dy \\ dz = a\,dx + b\,dy \end{cases}$$

where *dx* and *dy* can be isolated:

$$dx = \frac{y - b\,z}{a\,y - b\,x}\,dz \qquad\qquad dy = \frac{a\,z - x}{a\,y - b\,x}\,dz$$

Then the differential of the arc length of this hyperbola is:

$$ds = dx\,e_1 + dy\,e_2 + dz\,e_3 = \left(\frac{y - b\,z}{a\,y - b\,x}\,e_1 + \frac{a\,z - x}{a\,y - b\,x}\,e_2 + e_3 \right) dz$$

and its square: $$ds^2 = -dx^2 - dy^2 + dz^2 = \frac{1 - a^2 - b^2}{(a\,y - b\,x)^2}\,dz^2$$

Now we obtain the unitary vector *t* tangent to the hyperbola:

$$t = \frac{dx}{ds}\,e_1 + \frac{dy}{ds}\,e_2 + \frac{dz}{ds}\,e_3 = \frac{(y - b\,z)\,e_1 + (a\,z - x)\,e_2 + (a\,y - b\,x)\,e_3}{\sqrt{1 - a^2 - b^2}}$$

Observe that *t* and *ds* have imaginary components since a plane only cuts the two-sheeted hyperboloid if $a^2 + b^2 > 1$. Anyway, the geometric vector can be taken with real components, although then its modulus is –1.

The derivative of the tangent vector with respect to the arc length is not only proportional but equal to the normal vector *n* of the hyperbola:

$$n = \frac{dt}{ds} = -(x\,e_1 + y\,e_2 + z\,e_3) \qquad\qquad \Rightarrow \qquad \kappa = \left|\frac{dt}{ds}\right| = 1$$

because the position vector of every point on the two sheet hyperboloid has unitary modulus. Hence the curvature κ of the hyperbola is constant and equal to 1. All the hyperbolas being intersections of the hyperboloid with planes passing through the origin and a given point on the hyperboloid have the same radius of curvature *r*:

$$r=\left(\frac{dt}{ds}\right)^{-1}=-(x\,e_1+y\,e_2+z\,e_3)$$

Two consequences are deduced from this fact: firstly their common radius of curvature is perpendicular to the surface, and secondly, these hyperbolas are geodesics of the hyperboloid. Because all the hyperbolas lie on planes passing through the origin, their central projections are straight lines. In other words, every geodesic hyperbola is projected into a segment on the Beltrami disk. Taking as equation of the line:

$$v=k\,u+l$$

with k and l constant, the substitution in ds gives[4]:

$$\frac{ds}{e_{123}}=\frac{\sqrt{1+k^2-l^2}}{-u^2\left(1+k^2\right)-2\,k\,l\,u+1-l^2}\,du$$

By integration we arrive at the following primitive:

$$\frac{s}{e_{123}}=\operatorname{arg\,tgh}\frac{u\left(1+k^2\right)+k\,l}{\sqrt{1+k^2-l^2}}+\text{const}=\frac{1}{2}\log\frac{u+\dfrac{k\,l+\sqrt{1+k^2-l^2}}{1+k^2}}{u+\dfrac{k\,l-\sqrt{1+k^2-l^2}}{1+k^2}}+\text{const}$$

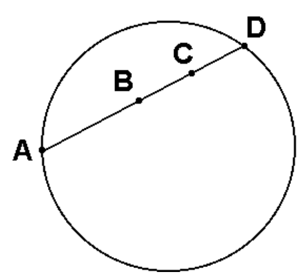
The line $v = k\,u + l$ cuts the Beltrami's circle $u^2+v^2=1$ at the points A and D whose coordinates are the solution of the system of the equations:

$$\begin{cases} v=k\,u+l \\ u^2+v^2=1 \end{cases} \Rightarrow \qquad u_A=\frac{-k\,l-\sqrt{1+k^2-l^2}}{k^2+1} \qquad u_D=\frac{-k\,l+\sqrt{1+k^2-l^2}}{k^2+1}$$

Thus the primitive may be rewritten:

$$\frac{s}{e_{123}}=\frac{1}{2}\log\frac{u-u_A}{u-u_D}+\text{const}$$

Figure 15.2



Then, the arc length between two points B and C on this geodesic hyperbola is the difference of this primitive between both points:

$$\frac{s_C-s_B}{e_{123}}=\frac{1}{2}\log\frac{(u_C-u_A)(u_D-u_B)}{(u_D-u_A)(u_C-u_B)}=\frac{1}{2}\log(A\,B\,C\,D)$$

[4] As said before, the arc length s on the two-sheeted hyperboloid has imaginary values. I'm sorry by the inconvenience of that reader accustomed to take real values. However the coherence of all the geometric algebra force to take account of the imaginary unity e_{123}. This also explains why Lobachevsky found the hyperboloidal trigonometry by choosing imaginary values for the sides in the spherical trigonometry.

that is, the half of the logarithm of the cross ratio of the four points (figure 15.2) on the Beltrami disk.

However it is more advantageous to write it using cosines instead of tangents by means of the trigonometric identity:

$$\cosh\frac{s_C - s_B}{e_{123}} \equiv \frac{1 - \mathrm{tgh}\,\frac{s_B}{e_{123}}\,\mathrm{tgh}\,\frac{s_C}{e_{123}}}{\sqrt{1 - \mathrm{tgh}^2\,\frac{s_B}{e_{123}}}\sqrt{1 - \mathrm{tgh}^2\,\frac{s_C}{e_{123}}}}$$

After removing *k* and *l* using the equation of the line in the primitive, we arrive at:

$$\cosh\frac{s_C - s_B}{e_{123}} = \frac{1 - u_B\, u_C - v_B\, v_C}{\sqrt{1 - u_B{}^2 - v_B{}^2}\sqrt{1 - u_C{}^2 - v_C{}^2}}$$

where the expression is real for all the points on the hyperboloid, whose projections lie on the Beltrami disk $u^2 + v^2 \leq 1$. This expression is equivalent to write:

$$\cosh\frac{s_C - s_B}{e_{123}} = \frac{(u_B\, e_1 + v_B\, e_2 + e_3)\cdot(u_C\, e_1 + v_C\, e_2 + e_3)}{|(u_B\, e_1 + v_B\, e_2 + e_3)|\;|(u_C\, e_1 + v_C\, e_2 + e_3)|}$$

Since the vector $u_A\, e_1 + v_A\, e_2 + e_3$ is proportional to the position vector, we have:

$$\cosh\frac{s_C - s_B}{e_{123}} = \cos(s_C - s_B) = \frac{(x_B\, e_1 + y_B\, e_2 + z_B\, e_3)\cdot(x_C\, e_1 + y_C\, e_2 + z_C\, e_3)}{|(x_B\, e_1 + y_B\, e_2 + z_B\, e_3)|\;|(x_C\, e_1 + y_C\, e_2 + z_C\, e_3)|}$$

which is the expected expression to calculate angles between vectors in the pseudo-Euclidean space. This result must be commented in more detail. Note that every pair of vectors going from the origin to the two-sheeted hyperboloid always lie on a central plane having hyperbolic nature, since it cuts the cone $z^2 - x^2 - y^2 = 0$ of vectors with zero length. Because of this, the angle measured on this plane is hyperbolic.

A hyperboloidal triangle is the region of the hyperboloid bounded by three geodesic hyperbolas. The length of a side of the triangle can be calculated by integration of the arc length of the hyperbola according to the last result. On the other hand, the angle of a vertex of a hyperboloidal triangle is the angle between the tangent vectors of the hyperbolas of each side at the vertex:

$$\cos\alpha = t\cdot t' = \frac{-(y - b\,z)(y - b'\,z) - (a\,z - x)(a'\,z - x) + (a\,y - b\,x)(a'\,y - b'\,x)}{\sqrt{1 - a^2 - b^2}\;\sqrt{1 - a'^2 - b'^2}}$$

$$= \frac{1 - a\,a' - b\,b'}{\sqrt{1 - a^2 - b^2}\;\sqrt{1 - a'^2 - b'^2}}$$

Note that every pair of tangent vectors lie on a plane of Euclidean nature, a plane parallel to a central plane not cutting the cone of zero length. So the angles measured on this plane are circular. We arrive at the same conclusion by proving that the absolute value of the cosine (and also its square) is lesser than the unity:

$$\frac{(1-a\,a'-b\,b')^2}{(1-a^2-b^2)^2\,(1-a'^2-b'^2)^2}<1$$

The denominator is positive (although both squares are negative because $a^2+b^2>1$) and it can pass to the right hand side of the inequality without changing the sense. Removing the denominator and after simplification:

$$-2\,a\,a'-2\,b\,b'+2\,a\,a'\,b\,b'<-a^2-b^2-a'^2-b'^2+a^2\,b'^2+a'^2\,b^2$$

and arranging terms we find an equivalent inequality:

$$a^2+a'^2-2\,a\,a'+b^2+b'^2-2\,b\,b'<a^2\,b'^2+a'^2\,b^2-2\,a\,a'\,b\,b'$$

$$(a-a')^2+(b-b')^2<(a\,b'-b\,a')^2$$

Just this is the condition for the existence of the intersection point where two central planes and the hyperboloid meet as I shall show now. The planes with equations $a\,x+b\,y=z$ and $a'\,x+b'\,y=z$ are projected into the lines $a\,u+b\,v=1$ and $a'\,u+b'\,v=1$ on the Beltrami disk. Then the point of intersection of both planes and the hyperboloidal surface is projected into the point on the Beltrami disk given by the system of equations:

$$\begin{cases} a\,u+b\,v=1 \\ a'\,u+b'\,v=1 \end{cases}$$

whose solution is: $$u=\frac{b'-b}{a\,b'-a'\,b} \qquad v=\frac{-a'+a}{a\,b'-a'\,b}$$

This point lies inside the Beltrami circle (and the point of intersection of the hyperboloid and both planes exists) if and only if:

$$u^2+v^2=\frac{(a-a')^2+(b-b')^2}{(a\,b'-b\,a')^2}<1$$

which is the condition above obtained.

Note that the expression for the angle between geodesic hyperbolas coincides with the angle between the bivectors of the planes containing these hyperbolas:

$$\cos\alpha=\frac{(a\,e_{23}+b\,e_{31}+e_{12})\bullet(a'\,e_{23}+b'\,e_{31}+e_{12})}{|\,a\,e_{23}+b\,e_{31}+e_{12}\,|\;|\,a'\,e_{23}+b'\,e_{31}+e_{12}\,|}$$

It is not a surprising result, because it happens likewise for the sphere: the angle between two maximum circles is the angle between the central planes containing them.

The planes passing through the origin only cut the two-sheeted hyperboloid if $a^2 + b^2 > 1$. These planes also cut the cone of zero length $z^2 - x^2 - y^2 = 0$ so they are hyperbolic and the angles measured on them are hyperbolic arguments. On the other hand, the planes with $a^2 + b^2 < 1$ do not cut the hyperboloid neither the cone and have Euclidean nature. In the Lobachevskian trigonometry, the planes containing the sides of a hyperboloidal triangle are always hyperbolic, while the planes touching the hyperboloid and containing the angles between sides are always Euclidean, and hence the angles are circular. Resuming, real modulus of a bivector ($0 < -a^2 - b^2 + 1$) indicates an Euclidean plane and imaginary modulus ($0 > -a^2 - b^2 + 1$) a hyperbolic plane.

Now we already have all the formulas to deduce the hyperboloidal trigonometry.

Hyperboloidal (Lobachevskian) trigonometry

The Lobachevskian trigonometry is the study of the trigonometric relations for the sides and angles of geodesic triangles on two-sheeted hyperboloids in the pseudo-Euclidean space. We will take for convenience the hyperboloid with unity radius $z^2 - x^2 - y^2 = 1$, although the trigonometric identities are equally valid for a hyperboloid with any radius.

Consider three points A, B and C on the unitary hyperboloid (figure 15.3). Then $|A| = |B| = |C| = 1$. The angles formed by each pair of sides will be denoted by α, β and γ, and the real value of the sides respectively opposite to these angles will be symbolised by a, b and c respectively. Then a is the real value of the arc of the geodesic hyperbola passing through the points B and C, that is, the hyperbolic angle between these vectors:

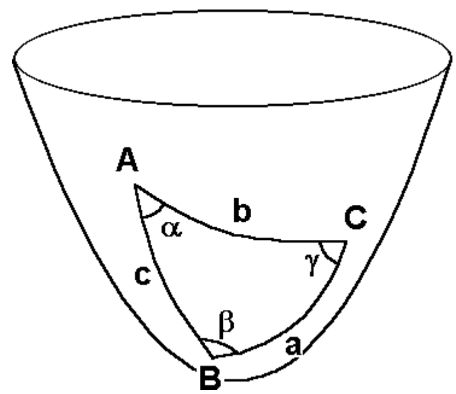


Figure 15.3

$$\sinh a = \frac{|B \wedge C|}{e_{123}}$$

As said above, the planes intersecting the hyperboloid have imaginary modulus. So we must divide by the pseudoscalar imaginary unity, which is also the volume element. In this equality the hyperbolic sine is taken positive and also a.

α is the circular angle between the sides b and c of the triangle, that is, the angle between the planes passing through the origin, A and B, and the origin, A and C respectively. Since the direction of a plane is given by its bivector, which can be obtained through the outer product, we have:

$$\sin\alpha = -\frac{|(A\wedge B)\times(A\wedge C)|}{|A\wedge B||A\wedge C|}$$

where the minus sign is due to the imaginary modulus of the outer products of the denominators. Now we write the products of the numerator using the geometric product:

$$\sin\alpha = -\frac{|-(A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)|}{8|A\wedge B||A\wedge C|}$$

$$\sin\alpha = -\frac{\left|-A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C - B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A\right|}{8|A\wedge B||A\wedge C|}$$

We extract the vector A as common factor at the left, but without writing it because $|A| = 1$:

$$\sin\alpha = -\frac{\left|-B\,A\,C + B\,C\,A + A\,B\,C - A^{-1}\,B\,A\,C\,A + C\,A\,B - C\,B\,A - A\,C\,B + A^{-1}\,C\,A\,B\,A\right|}{8|A\wedge B||A\wedge C|}$$

Applying the permutative property to the suitable pairs of products, we have:

$$\sin\alpha = -\frac{\left|6\,A\wedge B\wedge C + 2\,A^{-1}\,A\wedge B\wedge C\,A\right|}{8|A\wedge B||A\wedge C|} = -\frac{|A\wedge B\wedge C|}{|A\wedge B||A\wedge C|}$$

since the volume $A \wedge B \wedge C$ is a pseudoscalar, which commutes with all the elements of the algebra. Now the *law of sines for hyperboloidal triangles* follows:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c} = \frac{|A\wedge B\wedge C|}{|A\wedge B||B\wedge C||C\wedge A|\,e_{123}} \qquad \text{(I)}$$

Let us see the law of cosines. Since $e_{23}^2 = e_{31}^2 = -e_{12}^2 = 1$ then:

$$\cos\alpha = \frac{(A\wedge B)\bullet(A\wedge C)}{|A\wedge B||A\wedge C|} = -\frac{(A\wedge B)\bullet(A\wedge C)}{\sinh c\,\sinh b}$$

$$\sinh b\,\sinh c\,\cos\alpha = \frac{1}{8}\left((A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)\right)$$

$$= \frac{1}{8}\left(A\,B\,A\,C - A\,B\,C\,A - B\,A^2\,C + B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A\right)$$

Now taking into account that:

$$4\, A\cdot B\; A\cdot C = (A\,B + B\,A)(A\,C + C\,A) = A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C + B\,A\,C\,A$$

But also:

$$4\, A\cdot B\; A\cdot C = (A\,C + C\,A)(A\,B + B\,A) = A\,C\,A\,B + A\,C\,B\,A + C\,A^2\,B + C\,A\,B\,A$$

and adding the needed terms, we find:

$$\sinh b \sinh c \cos\alpha = \frac{1}{8}\left(8\, A\cdot B\; A\cdot C - 2\,A\,B\,C\,A - 2\,B\,A^2\,C - 2\,A\,C\,B\,A - 2\,C\,A^2\,B\right)$$

Extracting common factors and using $A^2 = B^2 = 1$, we may write:

$$\sinh b \sinh c \cos\alpha = \frac{1}{8}\left(8\, A\cdot B\; A\cdot C - 2\,A\,(B\,C + C\,B)\,A - 2\,(B\,C + C\,B)\right) = A\cdot B\; A\cdot C - B\cdot C$$

$$\sinh b \sinh c \cos\alpha = \cosh c \cosh b - \cosh a$$

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha \qquad \text{(II)}$$

which is the *law of cosines for sides*. The substitution of cosh c by means of the law of cosines gives:

$$\cosh a = \cosh b\,(\cosh a \cosh b - \sinh a \sinh b \cos\gamma) - \sinh b \sinh c \cos\alpha$$

$$\cosh a\,(1 - \cosh^2 b) = -\cosh b \sinh a \sinh b \cos\gamma - \sinh b \sinh c \cos\alpha$$

and the simplification of –sinh b:

$$\cosh a \sinh b = \cosh b \sinh a \cos\gamma + \sinh c \cos\alpha$$

The substitution of sinh c = sinh a sin γ / sin α yields:

$$\cosh a \sinh b = \cosh b \sinh a \cos\gamma + \frac{\sinh a \sin\gamma \cos\alpha}{\sin\alpha}$$

Dividing by sinh a :

$$\coth a \sinh b = \cosh b \cos\gamma + \sin\gamma \cot\alpha \qquad \text{(III)}$$

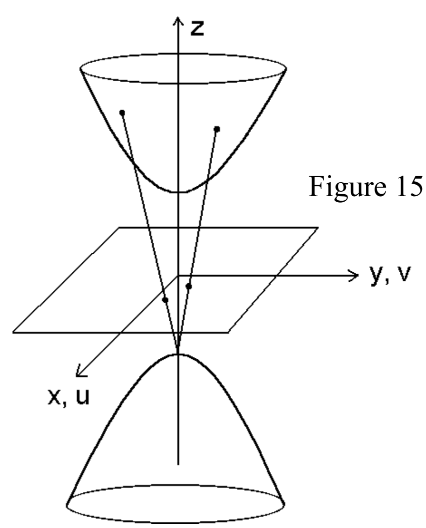


Figure 15.4

Stereographic projection (Poincaré disk)

We project the hyperboloidal surface into the plane $z = 0$ (figure 15.4) taking as centre of projection the lower pole (0, 0, –1). The upper sheet is projected inside the circle of unity radius while the lower sheet is projected outside. Let u and v be the

Cartesian coordinates of the projection plane. By similar triangles we have:

$$\frac{x}{u} = z + 1 \qquad \frac{y}{v} = z + 1$$

where (x, y, z) are the coordinates of a point on the hyperboloid. Using its equation $z^2 - x^2 - y^2 = 1$, one arrives at:

$$x = \frac{2\,u}{1 - u^2 - v^2} \qquad y = \frac{2\,v}{1 - u^2 - v^2} \qquad z = \frac{2}{1 - u^2 - v^2} - 1$$

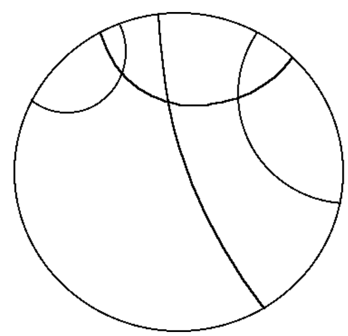
from where the differential of the arc length is obtained:

$$ds^2 = -dx^2 - dy^2 + dz^2 = -\frac{4\left(du^2 + dv^2\right)}{\left(1 - u^2 - v^2\right)^2}$$

The geodesic hyperbolas are intersection of the hyperboloid with central planes having the equation:

$$z = a\,x + b\,y$$

Figure 15.5



The substitution by the stereographic coordinates yields:

$$\left(u - a\right)^2 + \left(v - b\right)^2 = a^2 + b^2 - 1$$

which is the equation of a circle centred at (a, b) with radius $r = \sqrt{a^2 + b^2 - 1}$. That is, the geodesic hyperbolas on the hyperboloid are shown as circles in the stereographic projection (figure 15.5). The central planes cutting the two-sheeted hyperboloid fulfil $a^2 + b^2 > 1$, so that the radius is real and the centres of these circles are always placed outside the Poincaré disk.

The angle α between two circles is obtained through:

$$\cos\alpha = \frac{r^2 + r'^2 - \left(O - O'\right)^2}{2\,r\,r'} = \frac{a\,a' + b\,b' - 1}{\sqrt{a^2 + b^2 - 1}\sqrt{a'^2 + b'^2 - 1}}$$

Just this is the angle between two geodesic hyperbolas (between the tangent vectors t and t') found above. Therefore, the stereographic projection (Poincaré disk) is a conformal projection of the hyperboloidal surface.

On the other hand, the geodesic circles are always orthogonal to the Poincaré's circle $u^2 + v^2 = 1$ (r'=1, O' = (0, 0)) because:

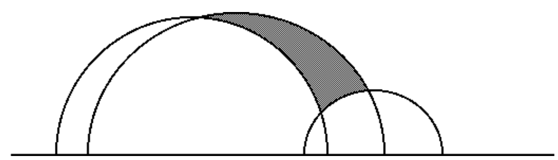
$$\cos\alpha = \frac{r^2 + 1 - O^2}{r} = 0$$

The differential of area is easily obtained through the modulus of the differential bivector:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{4\, du \wedge dv}{\left(1 - u^2 - v^2\right)^2}$$

When doing an inversion of the Poincaré disk centred at a point lying on the limit circle, another projection of the Lobachevsky's geometry is obtained: the Poincaré's half plane. Here the geodesics are semicircles orthogonal to the base line of the half plane. Since the inversion is a conformal transformation, the upper half plane is also a conformal projection of the hyperboloid. The figure 15.6 displays the projection of a hyperboloidal triangle and its sides in the Poincaré's upper half plane.

Figure 15.6



Azimuthal equivalent projection

This projection preserves the area and is similar to the azimuthal equivalent projection of the sphere usually used in the polar maps of the Earth.

The differential of the area in the pseudo-Euclidean space is a bivector whose square is:

$$dA^2 = (dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2$$

The substitution of the hyperboloid equation:

$$z^2 - x^2 - y^2 = 1 \qquad \Rightarrow \qquad dz = \frac{x}{z}dx + \frac{y}{z}dy$$

yields: $$dA^2 = \frac{1}{z^2}(dx \wedge dy)^2$$

Every azimuthal projection has plane coordinates u and v proportional to x and y, being the proportionality constant only function of z:

$$u = x\, f(z) \qquad v = y\, f(z)$$

Then by differentiation of these equalities and substitution of dz, which is a linear combination of dx and dy we arrive at:

$$du \wedge dv = \left(f^2 + \frac{z^2 - 1}{z} f\, f' \right) dx \wedge dy$$

Identifying both area differentials we find the following differential equation:

$$f^2 + \frac{z^2-1}{z} f\, f' = \frac{1}{z}$$

I introduce the auxiliary function $g = f^2$:

$$g + \frac{z^2-1}{2\,z} g' = \frac{1}{z}$$

Rewriting the equation with differentials, we arrive at an exact differential:

$$(z\,g-1)\,dz + \frac{z^2-1}{2} dg = 0 \quad \Leftrightarrow \quad d\left(\frac{z^2-1}{2} g - z\right) = 0$$

$$g(z) = \frac{2\,(z+\text{const})}{\text{z}^2-1} \quad \Rightarrow \quad f(z) = \sqrt{\frac{2\,(z+\text{const})}{\text{z}^2-1}}$$

Next to the pole x and u are coincident and also y and v. Then $f(1)=1$, which implies the integration constant be -1 and then[5]:

$$f(z) = \sqrt{\frac{2\,(z-1)}{\text{z}^2-1}} = \sqrt{\frac{2}{z+1}} \quad \Rightarrow \quad u = x\sqrt{\frac{2}{z+1}} \qquad v = y\sqrt{\frac{2}{z+1}}$$

Weierstrass coordinates and cylindrical equidistant projection

These are a set of coordinates similar to the spherical coordinates, but on the hyperboloid surface in the pseudo-Euclidean space.

For a hyperboloid with unity radius, the Weierstrass coordinates are:

$$x = \sinh\psi\,\cos\varphi$$

$$y = \sinh\psi\,\sin\varphi$$

$$z = \cosh\psi$$

Then the differential of arc length is:

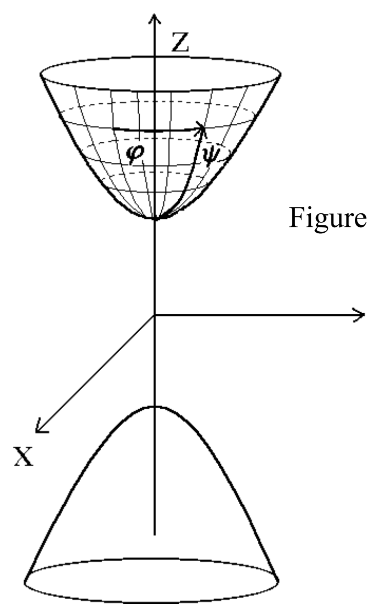


Figure 15.7

[5] As a comparison in the central projection $f(z) = 1/z$ while for the stereographic projection $f(z) = 1\,/\,(z+1)$.

$$ds = (\cosh\psi \cos\varphi \, d\psi - \sinh\psi \sin\varphi \, d\varphi) e_1 + (\cosh\psi \sin\varphi \, d\psi + \sinh\psi \cos\varphi \, d\varphi) e_2 + \sinh\psi \, d\psi \, e_3$$

Introducing the unitary vectors e_ψ and e_φ as:

$$e_\psi = \cosh\psi \cos\varphi \, e_1 + \cosh\psi \sin\varphi \, e_2 + \sinh\psi \, e_3$$

$$e_\varphi = -\sin\varphi \, e_1 + \cos\varphi \, e_2$$

the differential of arc length in hyperboloidal coordinates becomes:

$$ds = d\psi \, e_\psi + \sinh\psi \, d\varphi \, e_\varphi \qquad\qquad ds^2 = -\left(d\psi^2 + \sinh^2\psi \, d\varphi^2\right)$$

Note that e_ψ and e_φ are orthogonal vectors, since their inner product is zero. At $\psi = 0$, ds only depends on $d\psi$, so that there is a pole. Then ψ is the arc length from the pole to the given point on the hyperboloid surface (figure 15.7), while φ is the arc length over the equator, the parallel determined by $\sinh\psi = 1$ and $\psi = \log(1+\sqrt{2})$. The meridians (φ = constant) are geodesics, while the parallels (ψ = constant) including the equator are not geodesics because their planes do not pass through the origin of coordinates.

In the cylindrical projections, the hyperboloid is projected onto a cylinder passing through its equator, whose axis is the *z* axis. The chart is the Euclidean map obtained unrolling the cylinder[6]. Using φ and ψ as the coordinates *u* and *v* of the chart, the hyperboloid surface is projected into a rectangle with 2π width and infinite height. The area on this chart is:

$$dA = \sinh\psi \, d\varphi \wedge d\psi = \sinh v \, du \wedge dv$$

This projection is equidistant for the meridians (φ = constant) and for the equator ($\psi = \log(1+\sqrt{2})$).

Cylindrical conformal projection

If we wish to preserve the angles between curves, we must enlarge the meridians by the same amount as the parallels are enlarged in a cylindrical projection, that is by a factor $1/\sinh\psi$:

$$dv = \frac{d\psi}{\sinh\psi} \quad \Rightarrow \quad v = \log \operatorname{tgh}\frac{\psi}{2}$$

The differential of arc length and area are:

---

[6] Every plane tangent to the two-sheeted hyperboloid has Euclidean nature. So the Lobachevsky's surface can be only projected onto a Euclidean chart, and the cylinder of projection is not hyperbolic and does not belong properly to the pseudo-Euclidean space.

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi\, d\varphi^2\right) = -\sinh^2\psi\left(du^2 + dv^2\right) = \frac{-4\exp(2v)}{\left(1-\exp(2v)\right)^2}\left(du^2 + dv^2\right)$$

$$dA = \sinh\psi\, d\varphi \wedge d\psi = \frac{4\exp(2v)}{\left(1-\exp(2v)\right)^2}\, du \wedge dv$$

Cylindrical equivalent projection

If we wish to preserve area, we must shorten the meridians in the same amount as the parallels are enlarged in the cylindrical projection, what yields:

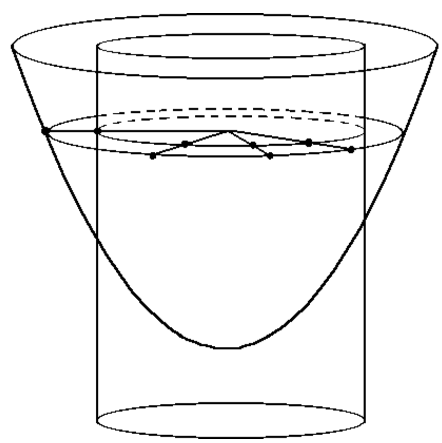
$$dv = \sinh\psi\, d\psi \qquad \Rightarrow \qquad v = \cosh\psi$$

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi\, d\varphi^2\right) = -\left(v^2 - 1\right)du^2 - \frac{1}{v^2 - 1}dv^2$$

$$dA = \sinh\psi\, d\varphi \wedge d\psi = du \wedge dv$$

which clearly displays the equivalence of the projection. Observe that $v = \cosh\,\psi$ means the hyperboloid is projected following planes perpendicular to the cylinder of projection (figure 15.8).

Figure 15.8



Conic projections

These projections are made into a cone surface tangent to the hyperboloid. Here to take the cone $z^2 - x^2 - y^2 = 0$ is the most natural choice, although as before, the cone of projection has Euclidean nature and does not belong properly to the pseudo-Euclidean space, where it should be the cone of zero length. The cone surface unrolled is a circular sector. In a conic projection a parallel is shown as a circle with zero distortion. The characteristic parameter of a conic projection is the constant of the cone $n = \cos\,\theta_0$, being θ_0 the Euclidean angle of inclination of the generatrix of the cone. The relation with the hyperbolic angle ψ_0 of the parallel touching the cone (which is represented without distortion in the projection) and θ_0 is:

$$n = \cos\theta_0 = \frac{1}{\sqrt{1 + \mathrm{tgh}^2\,\psi_0}}$$

Since the graticule of the conic projections is radial, is more convenient to use the radius r and the angle χ :

$$dr = f(\psi)\, d\psi \qquad\qquad d\chi = n\, d\varphi$$

The differentials of arc length and area for a conic projection are:

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi\, d\varphi^2\right) = -\left(\frac{dr^2}{[f(\psi)]^2} + \frac{\sinh^2\psi}{n^2}\, d\chi^2\right)$$

$$dA = \sinh\psi\, d\varphi \wedge d\psi = \frac{\sinh\psi}{n\, f(\psi)}\, dr \wedge d\chi$$

Let us see as before the three special cases: equidistant, conformal and equivalent projections. The differential of area for polar coordinates r, χ is $dA = r\, dr \wedge d\chi$. If the projection is equivalent, we must identify both dA to find:

$$\frac{d}{d\psi}\left[\frac{\sinh\psi}{n\, f(\psi)}\right] = f(\psi) \qquad \Rightarrow \qquad f(\psi) = \frac{1}{\sqrt{n}}\cosh\frac{\psi}{2} \quad \text{and} \quad r = \frac{2}{\sqrt{n}}\sinh\frac{\psi}{2}$$

$$ds^2 = -\left(\frac{n}{1+\dfrac{n\, r^2}{4}}\, dr^2 + \frac{1+\dfrac{n\, r^2}{4}}{n}\, r^2 d\chi^2\right)$$

If the projection is equidistant, the meridians have zero distortion so $d\psi = dr / n$ and:

$$ds^2 = -\frac{1}{n^2}\left(dr^2 + \sinh^2\left(\frac{r}{n}\right) d\chi^2\right)$$

If the projection is conformal then $ds^2 \propto dr^2 + r^2 d\chi^2$ so:

$$ds^2 = -\frac{\sinh^2\psi}{n^2 r^2}\left(dr^2 + r^2\, d\chi^2\right)$$

Then we solve the differential equation:

$$d\psi = \frac{\sinh\psi}{n\, r}\, dr$$

with the boundary condition $\operatorname{tgh}\psi_0 = r_0$ to find the conformal projection:

$$r = \operatorname{tgh}\psi_0 \left(\frac{\operatorname{tgh}\dfrac{\psi}{2}}{\operatorname{tgh}\dfrac{\psi_0}{2}} \right)^n$$

On the congruence of geodesic triangles

Two geodesic triangles on the hyperboloid having the same angles also have the same sides and are said to be *congruent*. This follows immediately from the rotations in the pseudo-Euclidean space, which are expressed by means of tetranions in the same way as quaternions are used in the rotations of Euclidean space. However, this falls out of the scope of this book and will not be treated. The reader must perceive, in spite of his Euclidean eyes[7], that all the points on the hyperboloid are equivalent because the curvature is always the unity and the surface is always perpendicular to the radius. So the pole (vertex of the hyperboloid) is not any special point, and any other point may be chosen as a new pole provided that the new axis are obtained from the old ones through a rotation.

Comment about the names of the non-Euclidean geometry

Finally it is apparent that the different kinds of non-Euclidean geometry have been often misnamed. The Lobachevskian surface was improperly called the “hyperbolic plane” in contradiction with the fact that it has constant curvature and therefore is not flat. Certainly, in the Lobachevsky’s geometry the hyperbolic functions are widely used, but in the same way as the circular functions work in the spherical trigonometry, as Lobachevsky himself showed. In the pseudo-Euclidean plane the geometry of the hyperbola is properly realised, the hyperbolic functions being also defined there. Thus the pseudo-Euclidean plane should be named properly the *hyperbolic plane*, while the Lobachevsky’s geometry might be alternatively called *hyperboloidal geometry*. The plane models of the Lobachevsky’s geometry (Beltrami disk, Poincaré disk and half plane, etc.) must be understood as plane projections of the hyperboloid in the pseudo-Euclidean space, which is its proper space, and then the Lobachevskian trigonometry may be called *hyperboloidal trigonometry*.

On the other hand, the space-time was the first physical discovery of a pseudo-Euclidean plane, but one no needs to appeal to relativity because, as the lector has viewed in this book, both signatures of the metrics (Euclidean and pseudo-Euclidean) are included in the plane geometric algebra and also in the algebras of higher dimensions.

Exercises

15.1. According to the Lobachevsky’s axiom of parallelism, at least two parallel lines pass through an exterior point of a given line. The two lines approaching a given line at

[7] In fact, eyes are not Euclidean but a very perfect camera where images are projected on a spherical surface. The principles of projection are likewise applicable to the pseudo-Euclidean space. Our mind accustomed to the ordinary space (we learn its Euclidean properties in the first years of our life) deceives us when we wish perceive the pseudo-Euclidean space.

the infinity are called «parallel lines» while those not cutting and not approaching it are called «ultraparallel» or «divergent lines». Lobachevsky defined the angle of parallelism $\Pi(s)$ as the angle which forms the line parallel to a given line with its perpendicular, which is a function of the distance s between the intersections on the perpendicular. Prove that $\Pi(s) = 2$ arctg(exp($-s$)) using the stereographic projection.

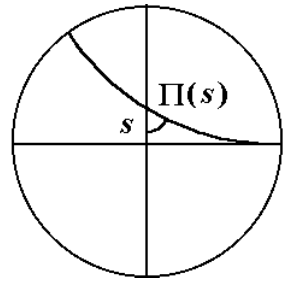


Figure 15.9

15.2. Find the equivalent of the Pythagorean theorem in the Lobachevskian geometry by taking a right angle triangle.

15.3. A *circle* is defined as the curve which is the intersection of the hyperboloid with a plane not passing through the origin with a slope less than the unity. Show that:

a) The curve obtained is an ellipse.

b) The distance from a point that we call the *centre* to every point of this ellipse is constant, and may be called the *radius of the circle*.

c) This ellipse is projected as a circle in the stereographic projection.

15.4. A *horocycle* is the intersection of the hyperboloid and a plane with unity slope ($a^2 + b^2 = 1$). Show that the horocycles are projected as circles touching the limit circle $x^2 + y^2 = 1$ in the stereographic projection. Find the centre of a horocycle.

15.5 a) Show that the differential of area in the Beltrami projection is:

$$dA = \frac{du \wedge dv}{\left(1 - u^2 - v^2\right)^{3/2}}$$

b) By integrating dA prove that the area between two lines forming an angle φ and the common "parallel" line (figure 15.10-a) is equal to $\pi - \varphi$.

c) Prove that the area of every triangle (figure 15.10-b) is equal to its angular defect $\pi - \alpha - \beta - \gamma$.

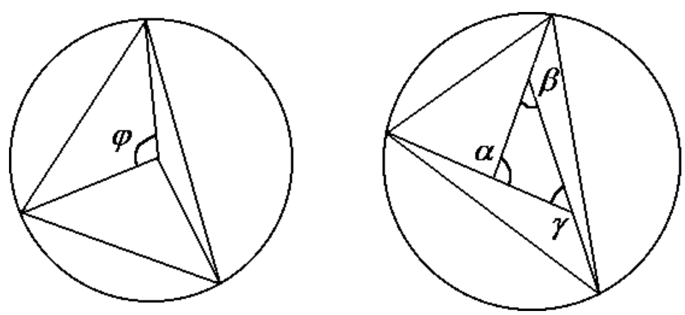


Figure 15.10

15.6 Use the azimuthal conformal projection to show that the area of a circle (hyperboloidal segment) with radius ψ is equal to $2\pi(\cosh\psi - 1)$. This result is analogous to the area of a spherical segment $2\pi(1 - \cos\theta)$.

16. SOLUTIONS OF THE PROPOSED EXERCISES

1. The vectors and their operations

1.1 Let a, b be the different sides of the parallelogram and c, d both diagonals. Then:

$$c = a + b \qquad\qquad d = a - b$$

$$c^2 + d^2 = (a + b)^2 + (a - b)^2 = a^2 + 2\, a \cdot b + b^2 + a^2 - 2\, a \cdot b + b^2 =$$

$$= 2\, a^2 + 2\, b^2$$

which are the sum of the squares of the four sides of the parallelogram.

1.2 $$(a \cdot b)^2 - (a \wedge b)^2 = \frac{(a\, b + b\, a)^2}{4} + \frac{(a\, b - b\, a)^2}{4}$$

$$= \frac{a\, b\, a\, b + b\, a\, b\, a + a\, b^2\, a + b\, a^2\, b}{4} - \frac{a\, b\, a\, b + b\, a\, b\, a - a\, b^2\, a - b\, a^2\, b}{4}$$

$$= a^2\, b^2$$

1.3 Let us prove that $a \wedge b\;\; c \wedge d = (a \wedge b\;\; c) \cdot d$:

$$4\, a \wedge b\;\; c \wedge d = (a\, b - b\, a)\,(c\, d - d\, c) = a\, b\, c\, d - a\, b\, d\, c - b\, a\, c\, d + b\, a\, d\, c =$$

$$= a\, b\, c\, d - d\, b\, a\, c - b\, a\, c\, d + d\, a\, b\, c = (a\, b\, c - b\, a\, c)\, d + d\,(a\, b\, c - b\, a\, c) =$$

$$= 2\,(a\, b\, c - b\, a\, c) \cdot d = 4\,(a \wedge b\;\; c) \cdot d$$

In the same way the identity $a \wedge b\;\; c \wedge d = \mathrm{a} \cdot (b\;\; c \wedge d)$ is proved.

1.4 Using the identity of the previous exercise we have:

$$a \wedge b\;\; c \wedge d + a \wedge c\;\; d \wedge b + a \wedge d\;\; b \wedge c =$$

$$= a \cdot (b\;\; c \wedge d) + a \cdot (c\;\; d \wedge b) + a \cdot (d\;\; b \wedge c) =$$

$$= a \cdot (b\;\; c \wedge d + c\;\; d \wedge b + d\;\; b \wedge c) =$$

$$= a \cdot (b\, c\, d - b\, d\, c + c\, d\, b - c\, b\, d + d\, b\, c - d\, c\, b) = a \cdot 0 = 0$$

where the summands are simplified taking into account the permutative property.

1.5 Let us denote the components proportional and perpendicular to the vector a with $\|$ and $\perp$. Then using the fact that orthogonal vectors anticommute and those with the same direction commute, we have:

$$a\,b\,c = a\,(\,b_{\|} + b_{\perp}\,)\,(\,c_{\|} + c_{\perp}\,) = (\,b_{\|} - b_{\perp}\,)\,a\,(\,c_{\|} + c_{\perp}\,) = (\,b_{\|} - b_{\perp}\,)\,(\,c_{\|} - c_{\perp}\,)\,a =$$

$$= (\,b_{\|}\,c_{\|} - \,b_{\|}\,c_{\perp}\, - b_{\perp}\,c_{\|} + b_{\perp}\,c_{\perp}\,)\,a = (\,c_{\|}\,b_{\|} + c_{\perp}\,b_{\|} + c_{\|}\,b_{\perp} + c_{\perp}\,b_{\perp}\,)\,a =$$

$$= (\,c_{\|} + c_{\perp}\,)\,(\,b_{\|} + b_{\perp}\,)\,a = c\,b\,a$$

1.6 Let a, b, c be the sides of a triangle and h the altitude corresponding to the base b. Then the altitude divides the triangle in two right triangles where the Pythagorean theorem may be applied:

$$|\,b\,| = \sqrt{a^2 - h^2} + \sqrt{c^2 - h^2}$$

Isolating the altitude as function of the sides we obtain:

$$h^2 = a^2 - \frac{\left(a^2 + b^2 - c^2\right)^2}{4\,b^2}$$

Then the square of the area of the triangle is:

$$A^2 = \frac{b^2 h^2}{4} = \frac{4\,a^2 b^2 - \left(a^2 + b^2 - c^2\right)^2}{16} = \frac{-\,a^4 - b^4 - c^4 + 2\,a^2 b^2 + 2\,a^2 c^2 + 2\,b^2 c^2}{16}$$

Introducing the semiperimeter $s = \frac{1}{2}\,(\,|\,a\,| + |\,b\,| + |\,c\,|\,)$ we find:

$$A = \sqrt{s\,(s - |\,a\,|)\,(\,s - |\,b\,|)\,(\,s - |\,c\,|)}$$

2. A base of vectors for the plane

2.1 Draw, for example, two vectors of the first quadrant and mark u_1, u_2, v_1 and v_2 on the Cartesian axis. Then calculate the area of the parallelogram from the areas of the rectangles:

$$A = (\,u_1 + v_1\,)\,(\,u_2 + v_2\,) - u_1\,u_2 - v_1\,v_2 - 2\,v_1\,u_2 = u_1\,v_2 - u_2\,v_1$$

2.2 The area of a triangle is the half of the area of the parallelogram formed by any two sides. Therefore:

$$A = (\,3\,e_1 + 5\,e_2\,) \wedge (\,-2\,e_1 - 3\,e_2\,) = (\,-9 + 10\,)\,e_{12} = e_{12} \qquad \Rightarrow \qquad |\,A\,| = 1$$

2.3 $\quad a\,b\,c = (\,a_1\,e_1 + a_2\,e_2\,)\,(\,b_1\,e_1 + b_2\,e_2\,)\,(\,c_1\,e_1 + c_2\,e_2\,)$

$$= [\,a_1\,b_1 + a_2\,b_2 + e_{12}\,(\,a_1\,b_2 - a_2\,b_1\,)\,]\,(\,c_1\,e_1 + c_2\,e_2\,)$$

$$= e_1\,(a_1\,b_1\,c_1 + a_2\,b_2\,c_1 + a_1\,b_2\,c_2 - a_2\,b_1\,c_2) + e_2\,(a_1\,b_1\,c_2 + a_2\,b_2\,c_2 - a_1\,b_2\,c_1 + a_2\,b_1\,c_1)$$

$= (c_1\, e_1 + c_2\, e_2) [\, b_1\, a_1 + b_2\, a_2 + e_{12} (b_1\, a_2 - b_2\, a_1)] =$

$= (c_1\, e_1 + c_2\, e_2) (b_1\, e_1 + b_2\, e_2) (a_1\, e_1 + a_2\, e_2) = c\, b\, a$

2.4 $u\, v = (2\, e_1 + 3\, e_2) (-3\, e_1 + 4\, e_2) = 6 + 17\, e_{12}$

$|\, u\, v\, | = \sqrt{325}$ $\alpha(u, v) = 1.2315 = 70º33'36''$

2.5 For this base we have:

$e_1 \cdot e_2 = |\, e_1 \,|\,|\, e_2 \,| \cos \pi/3 = 1$ $|\, e_1 \wedge e_2 \,| = |\, e_1 \,|\,|\, e_2 \,|\,|\sin \pi/3\,| = \sqrt{3}$

Applying the original definitions of the modulus of a vector one finds:

$u^2 = (2\, e_1 + 3\, e_2)^2 = 4\, e_1^2 + 9\, e_2^2 + 12\, e_1 \cdot e_2 = 4 + 36 + 12 = 52 \quad \Rightarrow \quad |\, u\, | = \sqrt{52}$

$v^2 = (-3\, e_1 + 4\, e_2)^2 = 9\, e_1^2 + 16\, e_2^2 - 24\, e_1 \cdot e_2 = 9 + 64 - 24 = 49 \quad \Rightarrow \quad |\, v\, | = \sqrt{49}$

$u\, v = (2\, e_1 + 3\, e_2) (-3\, e_1 + 4\, e_2) = -6\, e_1^2 + 12\, e_2^2 + 8\, e_1\, e_2 - 9\, e_2\, e_1 =$

$= 42 - e_1 \cdot e_2 + 17\, e_1 \wedge e_2 = 41 + 17\, e_1 \wedge e_2 \quad \Rightarrow \quad |\, u\, v\, | = \sqrt{2548}$

$$\cos\alpha = \frac{u \cdot v}{|\, u\, v\, |} = \frac{41}{\sqrt{2548}} \qquad \sin\alpha = \pm\frac{|\, u \wedge v\, |}{|\, u\, v\, |} = \frac{17\sqrt{3}}{\sqrt{2548}}$$

$\alpha(u, v) = 0.6228 = 35º\ 41'\ 5''$ oriented with the sense from e_1 to e_2 .

When e_1 and e_2 are not perpendicular, $e_1\, e_2 \neq - e_2\, e_1$ and e_{12} has not the meaning of a pure area but the outer product $e_1 \wedge e_2$ with an area of $\sqrt{3}$. Also observe that $|\, u\, v\, | = |\, u\, |\,|\, v\, |$.

2.6 In order to find the components of v for the new base $\{u_1, u_2\}$, we must resolve the vector v into a linear combination of u_1 and u_2 .

$v = c_1\, u_1 + c_2\, u_2$

$$c_1 = \frac{v \wedge u_2}{u_1 \wedge u_2} = \frac{3(-3) - (-5)5}{-1} = -16 \qquad c_2 = \frac{u_1 \wedge v}{u_1 \wedge u_2} = \frac{2(-5) - (-1)3}{-1} = 7$$

so in the new base $v = (-16, 7)$.

3. The complex numbers

3.1 $z\, t = (1 + 3\, e_{12}) (-2 + 2\, e_{12}) = -2 - 6 + (2 - 6)\, e_{12} = -8 - 4\, e_{12}$

3.2 Every complex number z can be written as product of two vectors a and b, its modulus being the product of the moduli of both vectors:

$$z = a\,b \quad \Rightarrow \quad |z| = |a|\,|b| \quad \Rightarrow \quad |z|^2 = a^2\, b^2 = a\, b\, b\, a = z\, z^*$$

3.3 The equation $x^4 - 1 = 0$ is solved by extraction of the fourth roots:

$$x^4 = 1 \quad \begin{array}{l} x^2 = 1 \quad \Rightarrow \quad x_1 = 1,\ x_2 = -1 \\ x^2 = -1 \quad \Rightarrow \quad x_3 = e_{12}\,,\ x_4 = -e_{12} \end{array}$$

3.4 Passing to the polar form we have:

$$\left(\sqrt{2}_{\pi/4}\right)^n = -\left(\sqrt{2}_{-\pi/4}\right)^n$$

$$\sqrt{2^n}_{\,n\pi/4} = -\sqrt{2^n}_{\,-n\pi/4} \quad \Rightarrow \quad \sqrt{2^n}_{\,n\pi/4} = \sqrt{2^n}_{\,-n\pi/4+\pi}$$

Therefore the arguments must be equal except for k times 2π:

$$\frac{n\pi}{4} = -\frac{n\pi}{4} + \pi + 2k\pi \quad \Rightarrow \quad \frac{n\pi}{2} = \pi + 2k\pi \quad \Rightarrow \quad n = 2 + 4k$$

3.5 The three cubic roots of $-3 + 3\, e_{12}$ are $\sqrt[3]{18}_{\pi/4}$, $\sqrt[3]{18}_{11\pi/12}$, $\sqrt[3]{18}_{19\pi/12}$.

3.6 Using the formula of the equation of second degree we find: $z_1 = 2-3\, e_{12}$, $z_2 = 1+e_{12}$.

3.7 We suppose that the complex analytic extension f has a real part a of the form:

$$a = \sin x\, K(y) \qquad \text{with } K(0) = 1$$

Applying the first Cauchy-Riemann condition, we find the imaginary part b of f:

$$\frac{\partial a}{\partial x} = \cos x\, K(y) = \frac{\partial b}{\partial y} \quad \Rightarrow \quad b = \cos x \int K(y)\, dy$$

Applying the second Cauchy-Riemann condition:

$$\frac{\partial a}{\partial y} = \sin x\, K'(y) = -\frac{\partial b}{\partial x} = \sin x \int K(y)\, dy$$

we arrive at a differential equation for $K(y)$ whose solution is the hyperbolic cosine:

$$\left.\begin{array}{r} K'(y) = \int K(y)\, dy \\ K(0) = 1 \end{array}\right\} \quad \Rightarrow \quad K(y) = \cosh y$$

Hence the analytical extension of the sine is:

$$\sin(x + e_{12}\, y) = \sin x \cosh y + e_{12} \cos x \sinh y$$

The analytical extensions of the trigonometric functions may be also obtained from the exponential function. In the case of the cosine, we have:

$$\cos z = \frac{\exp(e_{12}\, z) + \exp(-e_{12}\, z)}{2} = \frac{\exp(-y)(\cos x + e_{12} \sin x) + \exp(y)(\cos x - e_{12} \sin x)}{2}$$

$$\cos z = \cos x \cosh y - e_{12} \sin x \sinh y$$

3.8 Let us calculate some derivatives at $z = 0$:

$$f(z) = \log(1 + \exp(-z)) \qquad f(0) = \log 2$$

$$f'(z) = \frac{-\exp(-z)}{1 + \exp(-z)} = \frac{-1}{\exp(z) + 1} \qquad f'(0) = -\frac{1}{2}$$

$$f''(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} \qquad f''(0) = \frac{1}{4}$$

$$f'''(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} - \frac{2\exp(2z)}{(\exp(z) + 1)^3} \qquad f'''(0) = 0$$

$$f^{IV}(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} - \frac{2\exp(2z)}{(\exp(z) + 1)^3} - \frac{4\exp(2z)}{(\exp(z) + 1)^3} + \frac{6\exp(3z)}{(\exp(z) + 1)^4} \qquad f^{IV}(0) = -\frac{1}{8}$$

Hence the Taylor series is:

$$f(z) = \log 2 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^4}{8} + \ldots$$

Observe that $f(e_{12}\pi) = \log(0)$ is divergent. This is the singularity nearest the origin. Therefore the radius of convergence of the series is π.

3.9 Separating fractions:

$$f(z) = \frac{1}{z^2 + 2z - 8} = \frac{1}{6(z-2)} - \frac{1}{6(z+4)}$$

and developing the second fraction:

$$\frac{1}{z+4}=\frac{1}{6+z-2}=\frac{\frac{1}{6}}{1+\frac{z-2}{6}}=\frac{1}{6}\left(1-\frac{z-2}{6}+\left(\frac{z-2}{6}\right)^2-\left(\frac{z-2}{6}\right)^3+\ldots\right)$$

we find: $$f(z)=\frac{1}{6(z-2)}-\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{6^{n+2}}$$

This Lauren series is convergent in the annulus $0<|z-2|<6$, which contains the required annulus $1<|z-2|<4$.

3.10 Taking into account that $(1-x)^{-1}=1+x+x^2+\ldots$, the function defined by the series is:

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{4^n(z+1)^n}=\frac{1}{1-\frac{1}{4(z+1)}}-1=\frac{1}{4z+3}$$

Now we see that at $z=-3/4$ the function has a pole. Therefore the radius of convergence of this series (centred at $z=-1$) is $|-3/4-(-1)|=1/4$.

3.11 From the successive derivatives of the sines calculated at $z=0$, one obtains the Taylor series for $\sin z$:

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

so the Lauren series for the given function is:

$$\frac{\sin z}{z^2}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n+1)!}=\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-\ldots$$

Since the pole $z=0$ is the unique singularity (see that the analytical extension of the real sine in the exercise 3.7 has no singularities), the annulus of convergence is $0<|z|<\infty$.

3.12 If $f(z)$ is analytic then the derivative at each point is unique and we can write for two different directions dz_1 and dz_2:

$$df_1=dz_1\,f'(z) \qquad\qquad df_2=dz_2\,f'(z)$$

According to the relationship between the complex and vectorial planes, we can multiply by e_1 at the left in order to turn the complex differentials into vectors:

$$df=da+db\,e_{12}\ \xrightarrow{e_1}\ df=da\,e_1+db\,e_2 \qquad\qquad dz=dx+dy\,e_{12}\ \xrightarrow{e_1}\ dz=dx\,e_1+dy\,e_2$$

Now let us calculate the geometric product of the vector differentials:

$$df_1\,df_2=dz_1\,f'(z)\,dz_2\,f'(z)=dz_1\,dz_2\,[f'(z)]^*\,f'(z)=|f'(z)|^2\,dz_1\,dz_2$$

Since $|f'(z)|^2$ is real, both geometric products have the same argument, $\alpha(df_1, df_2) = \alpha(dz_1, dz_2)$, and hence the transformation is conformal.

4. Transformations of vectors

4.1 If d is the direction vector of the reflection, the vector v' reflected of v with respect to this direction is given by:

$$v' = d^{-1}\, v\, d$$

If v'' is obtained from v' by a rotation through an angle α, and z is a unitary complex number with argument $\alpha/2$ then:

$$v'' = z^{-1}\, v'\, z \qquad\qquad z = \cos\frac{\alpha}{2} + e_{12}\,\sin\frac{\alpha}{2}$$

Joining both equations:

$$v'' = z^{-1}\, d^{-1}\, v\, d\, z = c^{-1}\, v\, c$$

That is, one obtains a reflection with respect another direction with vector $c = d\, z$, the product of the direction vector of the initial reflection and the complex number with half argument.

4.2 Let w'' be the transformed vector of w by two consecutive reflections with respect different directions u and v:

$$w'' = v^{-1}\, w'\, v = v^{-1}\, u^{-1}\, w\, u\, v$$

Then we can write:

$$w'' = z^{-1}\, w\, z \qquad\qquad z = u\, v \qquad\qquad z \text{ being a complex number}$$

so that it is equivalent to a rotation with an angle equal to the double of that formed by both direction vectors.

4.3 If the product of each transformed vector v' by the initial vector v is equal to a complex number z^2 (v' and v always form a constant angle) then:

$$v\, v' = z^2$$

$$v' = v^{-1}\, z^2 = z^*\, v^{-1}\, z = z^{-1}\, |\,z\,|^2\, v^{-1}\, z = |\,z\,|^2\, (\,z^{-1}\, v\, z\,)^{-1}$$

which represents an inversion with radius $|\,z\,|^2$ followed by a rotation with an angle equal to the double of the argument of z. As the algebra shows, both elemental transformations commute.

5. Points and straight lines

5.1 If A, B, C and D are located following this order on the perimeter of the parallelogram, then $AB=DC$:

$$AB = DC \quad \Rightarrow \quad B - A = C - D \quad \Rightarrow \quad D = C - B + A$$

$$D = (2, -5) - (4, -3) + (2, 4) = (0, 2)$$

The area is: $\quad |AB \wedge BC| = |(2\, e_1 - 7\, e_2) \wedge (-2\, e_1 - 2\, e_2)| = |-18\, e_{12}| = 18$

5.2 The Euler's theorem:

$$AD\, BC + BD\, CA + CD\, AB = (D - A)(C - B) + (D - B)(A - C) + (D - C)(B - A)$$

$$= C\,A - A\,C + A\,B - B\,A + B\,C - C\,B = 2\,(C \wedge A + A \wedge B + B \wedge C) =$$

$$= 2\,(B - A) \wedge (C - B) = 2\, AB \wedge BC$$

This product only vanishes if A, B and C are collinear. On the other hand we see that the product is the oriented area of the triangle ABC.

5.3 a) The area of the triangle ABC is the outer product of two sides:

$$AB = B - A = (4, 4) - (2, 2) = 2\, e_1 + 2\, e_2 \qquad AC = C - A = (4, 2) - (2, 2) = 2\, e_1$$

$$AB \wedge AC = (2\, e_1 + 2\, e_2) \wedge 2\, e_1 = 4\, e_2 \wedge e_1$$

$$|AB \wedge AC| = 4\,|e_2 \wedge e_1| = 4\,|e_2|\,|e_1|\,|\sin 60^\circ| = 4\sqrt{3}$$

b) The distance from A to B is the length of the vector AB, etc:

$$AB^2 = (2\, e_1 + 2\, e_2)^2 = 4\, e_1^2 + 4\, e_2^2 + 8\, e_1 \cdot e_2 = 4 + 16 + 16 \cos \pi/3 = 28$$

$$AC^2 = (2\, e_1)^2 = 4 \qquad BC^2 = (-2\, e_2)^2 = 4\, e_2^2 = 16$$

$$d(A, B) = |AB| = \sqrt{28} \qquad d(B, C) = |BC| = 4 \qquad d(C, A) = |CA| = 2$$

5.4 A side of the trapezoid is a vectorial sum of the other three sides:

$$AD = AB + BC + CD$$

$$AD^2 = (AB + BC + CD)^2 = AB^2 + BC^2 + CD^2 + 2\, AB \cdot BC + 2\, AB \cdot CD + 2\, BC \cdot CD$$

$$= AB^2 + BC^2 + CD^2 + 2\, AB \cdot BC - 2\,|AB|\,|CD| + 2\, BC \cdot CD$$

where the fact that AB and CD be vectors with the same direction and contrary sense has been taken into account. Arranging terms:

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |AB|\ |CD| = 2\ AB \cdot BC + 2\ BC \cdot CD$$

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |\mathrm{AB}|\ |\mathrm{CD}| = 2\ BC \cdot (\ AB + CD\ )$$

Without loss of generality we will suppose that $|AB| > |CD|$:

$$|AB + CD| = |AB| - |CD|$$

Since the angle *ABC* is the supplement of the angle formed by the vectors *AB* and *BC*, we have:

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |AB|\ |CD| = -2\ |BC|\ (\ |AB| - |CD|\ ) \cos ABC$$

whence we obtain the angle *ABC*:

$$\cos ABC = \frac{-AD^2 + AB^2 + BC^2 + CD^2 - 2|AB||CD|}{2\ |BC|\big(|AB| - |CD|\big)}$$

The trapezoid can only exist for the range $-1 < \cos ABC < 1$, that is:

$$2\ |BC|\ (\ |AB| - |CD|\ ) > -AD^2 + AB^2 + BC^2 + CD^2 - 2\ |AB|\ |CD| >$$

$$> -2\ |BC|\ (|AB| - |CD|)$$

5.5 $\quad R = (\ 1 - p - q\ )\ O + p\ P + q\ Q \qquad \Rightarrow \qquad RP = (\ 1 - p - q\ )\ OP + q\ QP$

$$\Rightarrow \qquad RP \wedge PQ = (\ 1 - p - q\ )\ OP \wedge PQ$$

whence it follows that: Area $RPQ = (\ 1 - p - q\ )$ Area OPQ

5.6 The direction vector of the straight line *r* is *AB*:

$AB = B - A = (5, 4) - (2, 3) = 3\ e_1 + e_2 \qquad AC = C - A = (1, 6) - (2, 3) = -\ e_1 + 3\ e_2$

The distance from the point *C* to the line *r* is:

$$d(C,r) = \frac{|AC \wedge AB|}{|AB|} = \frac{|(-e_1 + 3e_2) \wedge (3e_1 + e_2)|}{\sqrt{10}} = \sqrt{10}$$

The angle between the vectors *AB* and *AC* is deduced by means of the sine and cosine:

$$\cos\alpha = \frac{AB \cdot AC}{|AB||AC|} = 0 \qquad\qquad e_{12}\ \sin\alpha = \frac{AB \wedge AC}{|AB||AC|} = e_{12}$$

Therefore, $\alpha = \pi/2$. The angle between two lines is always comprised from $-\pi/2$ to $\pi/2$

because a rotation of 2π around the intersection point does not alter the lines. When the angle exceeds these boundaries, you may add or subtract π.

5.7 a) Three points D, E, F are aligned if they are linearly dependent, that is, if the determinant of the coordinates vanishes.

$$D = (1 - x_D - y_D)\, O + x_D\, P + y_D\, Q$$

$$E = (1 - x_E - y_E)\, O + x_E\, P + y_E\, Q$$

$$F = (1 - x_F - y_F)\, O + x_F\, P + y_F\, Q$$

$$\begin{vmatrix} 1 - x_D - y_D & x_D & y_D \\ 1 - x_E - y_E & x_E & y_E \\ 1 - x_F - y_F & x_F & y_F \end{vmatrix} = 0$$

where the barycentric coordinate system is given by the origin O and points P, Q (for example the Cartesian system is determined by $O = (0, 0)$, $P = (1, 0)$ and $Q = (0, 1)$). The transformed points D', E' and F' have the same coordinates expressed for the base O', P' and Q'. Then the determinant is exactly the same, so that it vanishes and the transformed points are aligned. Therefore any straight line is transformed into another straight line.
b) Let O', P' and Q' be the transformed points of O, P and Q by the given affinity:

$$O' = (o_1, o_2) \qquad\qquad P' = (p_1, p_2) \qquad\qquad Q' = (q_1, q_2)$$

and consider any point R with coordinates (x, y):

$$R = (x, y) = (1 - x - y)\, O + x\, P + y\, Q$$

Then R', the transformed point of R, is:

$$R' = (1 - x - y)\, O' + x\, P' + y\, Q' = (1 - x - y)\,(o_1, o_2) + x\,(p_1, p_2) + y\,(q_1, q_2) =$$

$$= (x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1,\quad x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2) = (x', y')$$

where we see that the coordinates x' and y' of R' are linear functions of the coordinates of R:

$$x' = x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1$$

$$y' = x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2$$

In matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} p_1 - o_1 & q_1 - o_1 \\ p_2 - o_2 & q_2 - o_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} o_1 \\ o_2 \end{pmatrix}$$

Because every linear (and non degenerate) mapping of coordinates can be written in this regular matrix form, now we see that it is always an affinity.

c) Let us consider any three non aligned points A, B, C and their coordinates:

$$A=\left(1-x_A-y_A\right)O+x_AP+y_AQ$$

$$B=\left(1-x_B-y_B\right)O+x_BP+y_BQ$$

$$C=\left(1-x_C-y_C\right)O+x_CP+y_CQ$$

In matrix form:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}=\begin{pmatrix} 1-x_A-y_A & x_A & y_A \\ 1-x_B-y_B & x_B & y_B \\ 1-x_C-y_C & x_C & y_C \end{pmatrix}\begin{pmatrix} O \\ P \\ Q \end{pmatrix}$$

A certain point D is expressed with coordinates whether for $\{ O, P, Q \}$ or $\{ A, B, C \}$:

$$D=(1-b-c)A+b\,B+c\,C=(1-x_D-y_D)O+x_D\,P+y_D\,Q$$

In matrix form:

$$D=\begin{pmatrix} 1-x_D-y_D & x_D & y_D \end{pmatrix}\begin{pmatrix} O \\ P \\ Q \end{pmatrix}=\begin{pmatrix} 1-b-c & b & c \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\begin{pmatrix} 1-x_D-y_D & x_D & y_D \end{pmatrix}=\begin{pmatrix} 1-b-c & b & c \end{pmatrix}\begin{pmatrix} 1-x_A-y_A & x_A & y_A \\ 1-x_B-y_B & x_B & y_B \\ 1-x_C-y_C & x_C & y_C \end{pmatrix}$$

which leads to the following system of equations:

$$\begin{cases} x_D=\left(1-b-c\right)x_A+b\,x_B+c\,x_C \\ y_D=\left(1-b-c\right)y_A+b\,y_B+c\,y_C \end{cases}$$

An affinity does not change the coordinates x, y of A, B, C and D, but only the point base -$\{O', P', Q'\}$ instead of $\{O, P, Q\}$-. Therefore the solution of the system of equations for b and c is the same. Then we can write:

$$D'=\left(1-b-c\right)A'+b\,B'+c\,C'$$

d) If the points D, E, F and G are the consecutive vertices in a parallelogram then:

$$DE=GF \;\Rightarrow\; G=D-E+F$$

The affinity preserves the coordinates expressed in any base $\{D, E, F\}$. Then the transformed points form also a parallelogram:

$$G' = D' - E' + F' \;\Rightarrow\; D'E' = G'F'$$

e) For any three aligned points D, E, F the single ratio r is:

$$DE\,DF^{-1} = r \qquad \Rightarrow \qquad DE = r\,DF \qquad \Rightarrow \qquad E = (1 - r)\,D + r\,F$$

The ratio r is a coordinate within the straight line DF and it is not changed by the affinity:

$$E' = (1 - r)\,D' + r\,F' \qquad \Rightarrow \qquad D'E'\,D'F'^{-1} = r$$

5.8 This exercise is the dual of the problem 2. Then I have copied and pasted it changing the words for a correct understanding.
a) Three lines D, E, F are concurrent if they are linearly dependent, that is, if the determinant of the dual coordinates vanishes:

$$D = \left(1 - x_D - y_D\right) O + x_D\, P + y_D\, Q$$

$$E = \left(1 - x_E - y_E\right) O + x_E\, P + y_E\, Q$$

$$F = \left(1 - x_F - y_F\right) O + x_F\, P + y_F\, Q$$

$$\begin{vmatrix} 1 - x_D - y_D & x_D & y_D \\ 1 - x_E - y_E & x_E & y_E \\ 1 - x_F - y_F & x_F & y_F \end{vmatrix} = 0$$

where the dual coordinate system is given by the lines O, P and Q. For example, the Cartesian system is determined by $O = [0, 0]$ (line $-x - y + 1 = 0$), $P = [1, 0]$ (line $x = 0$) and $Q = [0, 1]$ (line $y = 0$). The transformed lines D', E' and F' have the same coordinates expressed for the base O', P' and Q'. Then the determinant also vanishes and the transformed lines are concurrent. Therefore any pencil of lines is transformed into another pencil of lines.
b) Let O', P' and Q' be the transformed lines of O, P and Q by the given transformation:

$$O' = [o_1, o_2] \qquad P' = [p_1, p_2] \qquad Q' = [q_1, q_2]$$

and consider any line R with dual coordinates $[x, y]$:

$$R = [x, y] = (1 - x - y)\,O + x\,P + y\,Q$$

Then R', the transformed line of R, is:

$$R' = (1 - x - y)\,O' + x\,P' + y\,Q' = (1 - x - y)\,[\,o_1, o_2\,] + x\,[\,p_1, p_2\,] + y\,[\,q_1, q_2\,] =$$

$$= [\,x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1,\quad x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2\,] = [\,x', y'\,]$$

where we see that the coordinates x' and y' of R' are linear functions of the coordinates of R:

$$x' = x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1$$

$$y' = x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2$$

In matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_1 - o_1 & q_1 - o_1 \\ p_2 - o_2 & q_2 - o_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \end{bmatrix}$$

Because any linear mapping (non degenerate) of dual coordinates can be written in this regular matrix form, now we see that it is always an affinity.

c) Let us consider any three non concurrent lines A, B, C and their coordinates:

$$A = \left(1 - x_A - y_A\right) O + x_A P + y_A Q$$

$$B = \left(1 - x_B - y_B\right) O + x_B P + y_B Q$$

$$C = \left(1 - x_C - y_C\right) O + x_C P + y_C Q$$

In matrix form:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{bmatrix} \begin{bmatrix} O \\ P \\ Q \end{bmatrix}$$

A certain line D is expressed with dual coordinates whether for $\{O, P, Q\}$ or $\{A, B, C\}$:

$$D = \left(1 - b - c\right) A + b\,B + c\,C = \left(1 - x_D - y_D\right) O + x_D\, P + y_D\, Q$$

In matrix form:

$$D = \begin{bmatrix} 1 - x_D - y_D & x_D & y_D \end{bmatrix} \begin{bmatrix} O \\ P \\ Q \end{bmatrix} = \begin{bmatrix} 1 - b - c & b & c \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} 1 - x_D - y_D & x_D & y_D \end{bmatrix} = \begin{bmatrix} 1 - b - c & b & c \end{bmatrix} \begin{bmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{bmatrix}$$

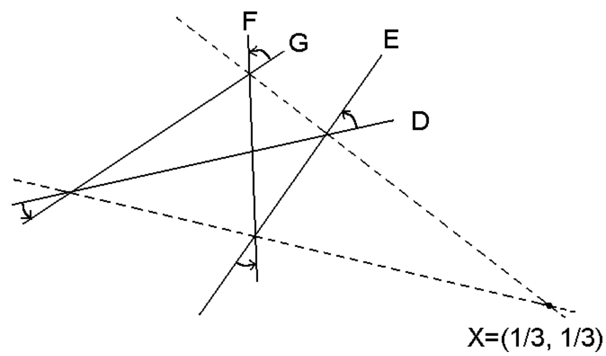
which leads to the following system of equations:

$$\begin{cases} x_D = (1-b-c)\,x_A + b\,x_B + c\,x_C \\ y_D = (1-b-c)\,y_A + b\,y_B + c\,y_C \end{cases}$$

An affinity does not change the coordinates x, y of A, B, C and D, but only the lines base -$\{O', P', Q'\}$ instead of $\{O, P, Q\}$-. Therefore the solution of the system of equations for b and c is the same. Then we can write:

$$D' = (1-b-c)\,A' + b\,B' + c\,C'$$

Figure 16.1



d) The affinity maps parallel points into parallel points (points aligned with the point (1/3, 1/3), point at the infinity in the dual plane). If the lines D, E, F and G are the consecutive vertices in a dual parallelogram (figure 16.1) then:

$$DE = GF \quad \Rightarrow \quad G = D - E + F$$

Where DE is the dual vector of the intersection point of the lines D and E, and GF the dual vector of the intersection point of G and F. Obviously, the points EF and DG are also parallel because from the former equality it follows:

$$EF = DG$$

The affinity preserves the coordinates expressed in any base $\{D, E, F\}$. Then the transformed lines form also a dual parallelogram:

$$G' = D' - E' + F' \quad \Rightarrow \quad D'E' = G'F'$$

e) For any three concurrent lines D, E, F the single dual ratio r is:

$$DE\,DF^{-1} = r \qquad \Rightarrow \qquad DE = r\,DF \qquad \Rightarrow \qquad E = (\,1-r\,)\,D + r\,F$$

The ratio r is a coordinate within the pencil of lines DF and it is not changed by the affinity:

$$E' = (\,1-r\,)\,D' + r\,F' \qquad \Rightarrow \qquad D'E'\,D'F'^{\,-1} = r$$

f) Let us consider four concurrent lines A, B, C and D with the following dual coordinates expressed in the lines base $\{O, P, Q\}$:

$$A = [x_A\,, y_A] \qquad B = [x_B\,, y_B] \qquad C = [x_C\,, y_C] \qquad D = [x_D\,, y_D]$$

Then the direction vector of the line *A* is:

$$v_A = (1 - x_A - y_A) v_O + x_A v_P + y_A v_Q$$

But the direction vectors of the base fulfils:

$$v_O + v_P + v_Q = 0$$

Then:

$$v_A = (-1 + 2x_A + y_A) v_P + (-1 + x_A + 2y_A) v_Q$$

And analogously:

$$v_B = (-1 + 2x_B + y_B) v_P + (-1 + x_B + 2y_B) v_Q$$

$$v_C = (-1 + 2x_C + y_C) v_P + (-1 + x_C + 2y_C) v_Q$$

$$v_D = (-1 + 2x_D + y_D) v_P + (-1 + x_D + 2y_D) v_Q$$

The outer product is:

$$v_A \wedge v_C = (x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A)) v_P \wedge v_Q$$

Then the cross ratio only depends on the dual coordinates but not on the direction of the base vectors (see chapter 10):

$$(ABCD) = \frac{v_A \wedge v_C \;\; v_B \wedge v_D}{v_A \wedge v_D \;\; v_B \wedge v_C} =$$

$$= \frac{(x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A))\,(x_D - x_B - (y_D - y_B) + 3(x_B\, y_D - x_D\, y_B))}{(x_D - x_A - (y_D - y_A) + 3(x_A\, y_D - x_D\, y_A))\,(x_C - x_B - (y_C - y_B) + 3(x_B\, y_C - x_C\, y_B))}$$

Hence it remains invariant under an affinity. Each outer product can be written as an outer product of dual vectors obtained by subtraction of the coordinates of each line and the infinite line:

$$v_A \wedge v_C = 3\left[x_A - \frac{1}{3}, y_A - \frac{1}{3}\right] \wedge \left[x_C - \frac{1}{3}, y_C - \frac{1}{3}\right] = 3\, LA \wedge LC$$

where $LA = A - L$ is the dual vector going from the line *L* at the infinity to the line *A*, etc. Then we can write this useful formula for the cross ratio of four lines:

$$(ABCD) = \frac{LA \wedge LC \;\; LB \wedge LD}{LA \wedge LD \; LB \wedge LC}$$

This exercise links with the section *Projective cross ratio* in the chapter 10.

5.9 To obtain the dual coordinates of the first line, solve the identity:

$$x - y - 1 \equiv a'(-x - y + 1) + b'x + c'y \qquad \Rightarrow \qquad a' = -1 \quad b' = 0 \quad c' = -2$$

$$\Rightarrow \qquad a = \frac{1}{3} \quad b = 0 \quad c = \frac{2}{3} \qquad \Rightarrow \qquad [0, 2/3]$$

For the second line:

$$x - y + 3 \equiv a'(-x - y + 1) + b'x + c'y \qquad \Rightarrow \qquad a' = 3 \quad b' = 4 \quad c' = 2$$

$$\Rightarrow \qquad a = \frac{3}{9} \quad b = \frac{4}{9} \quad c = \frac{2}{9} \qquad \Rightarrow \qquad [4/9, 2/9]$$

Both lines are aligned in the dual plane with the line at the infinity (whose dual coordinates are [1/3, 1/3]) since the determinant of the coordinates vanishes:

$$\begin{vmatrix} 1/3 & 0 & 2/3 \\ 2/9 & 4/9 & 2/9 \\ 1/3 & 1/3 & 1/3 \end{vmatrix} = 0$$

Therefore they are parallel (this is also trivial from the general equations).

5.10 The point (2, 1) is the intersection of the lines $x - 2 = 0$ and $y - 1 = 0$, whose dual coordinates are:

$$x - 2 = 0 \qquad \Rightarrow \qquad [1/5, 2/5]$$

$$y - 1 = 0 \qquad \Rightarrow \qquad [1/2, 0]$$

The difference of dual coordinates gives a dual direction vector for the point:

$$v = [1/2, 0] - [1/5, 2/5] = [3/10, -4/10]$$

and hence we obtain the dual continuous and general equations for the point:

$$\frac{b - 1/2}{3} = \frac{c}{-4} \qquad \Leftrightarrow \qquad 4b + 3c = 2$$

The point (–3, –1) is the intersection of the lines $x + 3 = 0$ and $y + 1 = 0$, whose dual coordinates are:

$$x + 3 = 0 \quad \Rightarrow \quad [4/10,\ 3/10]$$

$$y + 1 = 0 \quad \Rightarrow \quad [1/4,\ 2/4]$$

The difference of dual coordinates gives a dual direction vector for the point:

$$w = [1/4,\ 2/4] - [4/10,\ 3/10] = [-3/20,\ 4/20]$$

and hence we obtain the dual continuous and general equations for the point:

$$\frac{b - 1/4}{-3} = \frac{c - 2/4}{4} \quad \Leftrightarrow \quad 8b + 6c = 5$$

Note that both dual direction vectors v and w are proportional and the points are parallel in a dual sense. Hence the points are aligned with the centroid (1/3, 1/3) of the coordinate system.

6. Angles and elemental trigonometry

6.1 Let a, b and c be the sides of a triangle taken anticlockwise. Then:

$$a + b + c = 0 \quad \Rightarrow \quad c = -a - b \quad \Rightarrow \quad c^2 = (a + b)^2 = a^2 + b^2 + 2\, a \cdot b$$

$$c^2 = a^2 + b^2 + 2\,|a|\,|b|\,\cos(\pi - \gamma) = a^2 + b^2 - 2\,|\mathrm{a}|\,|\mathrm{b}|\,\cos\gamma \quad \text{(law of cosines)}$$

The area s of a triangle is the half of the outer product of any pair of sides:

$$2\,s = a \wedge b = b \wedge c = c \wedge a$$

$$|a|\,|b|\,\sin(\pi - \gamma)\, e_{12} = |b|\,|c|\,\sin(\pi - \alpha)\, e_{12} = |c|\,|a|\,\sin(\pi - \beta)\, e_{12}$$

$$\frac{\sin\gamma}{|c|} = \frac{\sin\alpha}{|a|} = \frac{\sin\beta}{|b|} \qquad \text{(law of sines)}$$

From here one quickly obtains:

$$\frac{|a| + |b|}{|a|} = \frac{\sin\alpha + \sin\beta}{\sin\alpha} \qquad \frac{|a| - |b|}{|b|} = \frac{\sin\alpha - \sin\beta}{\sin\beta}$$

By dividing both equations and introducing the identities for the addition and subtraction of sines we arrive at:

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}}{2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}$$

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\operatorname{tg}\dfrac{\alpha+\beta}{2}}{\operatorname{tg}\dfrac{\alpha-\beta}{2}} \qquad \text{(law of tangents)}$$

6.2 Let us substitute the first cosine by the half angle identity and convert the addition of the two last cosines into a product:

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=2\ \cos^2\frac{\alpha-\beta}{2}\ -1+\cos\frac{\beta-\alpha}{2}\cos\frac{\beta-2\gamma+\alpha}{2}$$

Let us extract common factor and convert the addition of cosines into a product:

$$=2\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha-2\gamma+\beta}{2}\right)-1$$

$$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha-\gamma}{2}\cos\frac{\gamma-\beta}{2}-1$$

$$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}-1$$

The identity for the sines is proved in a similar way.

6.3 Using the De Moivre's identity:

$$\cos 4\alpha+e_{12}\sin 4\alpha\equiv(\cos\alpha+e_{12}\sin\alpha)^4$$

After developing the right hand side we find:

$$\sin 4\alpha\equiv 4\cos^3\alpha\ \sin\alpha-4\cos\alpha\ \sin^3\alpha \qquad \cos 4\alpha\equiv\cos^4\alpha\ -6\cos^2\alpha\ \sin^2\alpha+\sin^4\alpha$$

And dividing both identities: $\operatorname{tg}4\alpha=\dfrac{4\operatorname{tg}\alpha-4\operatorname{tg}^3\alpha}{1-6\operatorname{tg}^2\alpha+\operatorname{tg}^4\alpha}$

6.4 Let α, β and γ be the angles of the triangle *PAB* with vertices *A*, *B* and *P* respectively. The angle γ embracing the arc *AB* is constant for any point *P* on the arc *AB*. By the law of sines we have:

$$\frac{|PA|}{\sin\alpha}=\frac{|PB|}{\sin\beta}=\frac{|PC|}{\sin\gamma}$$

The sum of both chords is:

$$|PA|+|PB|=|PC|\frac{\sin\alpha+\sin\beta}{\sin\gamma}$$

Converting the sum of sines into a product we find:

$$|PA|+|PB|=|PC|\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\gamma}=|PC|\frac{2\sin\frac{\pi-\gamma}{2}\cos\frac{\alpha-\beta}{2}}{\sin\gamma}$$

Since γ is constant, the maximum is attained for $\cos(\alpha/2-\beta/2)=1$, that is, when the triangle *PAB* is isosceles and *P* is the midpoint of the arc *AB*.

6.5 From the sine theorem we have:

$$\frac{|a|+|b|}{|c|}=\frac{\sin\alpha+\sin\beta}{\sin\gamma}=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}=\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}$$

Following the same way, we also have:

$$\frac{|a|-|b|}{|c|}=\frac{\sin\alpha-\sin\beta}{\sin\gamma}=\frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\gamma}{2}}$$

6.6 Take *a* as the base of the triangle and draw the altitude. It divides *a* in two segments which are the projections of the sides *b* and *c* on *a*:

$$|a|=|b|\cos\gamma+|c|\sin\beta \qquad \text{and so forth.}$$

6.7 From the double angle identities we have:

$$\cos\alpha\equiv\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\equiv1-2\sin^2\frac{\alpha}{2}\equiv2\cos^2\frac{\alpha}{2}-1$$

From the last two expressions the half-angle identities follow:

$$\sin\frac{\alpha}{2}\equiv\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2}\equiv\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Making the quotient:

$$\mathrm{tg}\,\alpha \equiv \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \equiv \frac{1-\cos\alpha}{\sin\alpha} \equiv \frac{\sin\alpha}{1+\cos\alpha}$$

7. Similarities and single ratio

7.1 First at all draw the triangle and see that the homologous vertices are:

$$P = (0, 0) \qquad P' = (4, 2)$$
$$Q = (2, 0) \qquad Q' = (2, 0)$$
$$R = (0, 1) \qquad R' = (5, 1)$$

$$PQ = 2\, e_1 \qquad QR = -2\, e_1 + e_2 \qquad RP = -e_2$$

$$P'Q' = -2\, e_1 + 2\, e_2 \qquad Q'R' = 3\, e_1 + e_2 \qquad R'P' = -e_1 + e_2$$

$$r = P'Q'\, PQ^{-1} = Q'R'\, QR^{-1} = R'P'\, RP^{-1} = -1 - e_{12} = \sqrt{2}_{\,5\pi/4}$$

The size ratio is $|r| = \sqrt{2}$ and the angle between the directions of homologous sides is $5\pi/4$.

7.2 Let *ABC* be a right triangle being *C* the right angle. The altitude *CD* cutting the base *AB* in *D* splits *ABC* in two right angle triangles: *ADC* and *BDC*. In order to simplify I introduce the following notation:

$$AB = c \qquad BC = a \qquad CA = b \qquad AD = x \qquad DB = c - x$$

The triangles *CBA* and *DCA* are oppositely similar because the angle *CAD* is common and the other one is a right angle. Hence:

$$b\, x^{-1} = (c\, b^{-1})^* = b^{-1}\, c \qquad \Rightarrow \qquad b^2 = c\, x$$

The triangles *DBC* and *CBA* are also oppositely similar, because the angle *DBC* is common and the other one is a right angle. Hence:

$$a\, (c - x)^{-1} = (c\, a^{-1})^* = a^{-1}\, c \qquad \Rightarrow \qquad a^2 = c\, (c - x)$$

Summing both results the Pythagorean theorem is obtained:

$$a^2 + b^2 = c\, (c - x) + c\, x = c^2$$

7.3 First at all we must see that the triangles *ABM* and *BCM* are oppositely similar. Firstly, they share the angle *BMC*. Secondly, the angles *MBC* and *BAM* are equal because they embrace the same arc *BC*. In fact, the limiting case of the angle *BAC* when *A* moves to *B* is the angle *MBA*. Finally the angle *BCM* is equal to the angle *ABM* because the sum of the angles of a triangle is π. The opposite similarity implies:

$$MA\, AB^{-1} = BC^{-1}\, MB \qquad \Rightarrow \qquad MA = BC^{-1}\, MB\, AB$$

$$MB\,AB^{-1} = BC^{-1}\,MC \qquad \Rightarrow \qquad MC^{-1} = AB\,MB^{-1}\,BC^{-1}$$

Multiplying both expressions we obtain:

$$MA\,MC^{-1} = BC^{-1}\,MB\,AB^2\,MB^{-1}\,BC^{-1} = AB^2\,BC^{-2}$$

7.4 Let Q be the intersection point of the segments PA and BC. The triangle PQC is directly similar to the triangle PBA and oppositely similar to the triangle BQA:

$$BA\,PA^{-1} = QC\,PC^{-1} \;\Rightarrow\; QC = BA\,PA^{-1}\,PC \;\Rightarrow\; |QC| = |BA|\;|PA|^{-1}\;|PC|$$

$$BQ\,BA^{-1} = PA^{-1}\,PB \;\Rightarrow\; BQ = PA^{-1}\,PB\,BA \;\Rightarrow\; |BQ| = |PA|^{-1}\;|PB|\;|BA|$$

Summing both expressions: $|BC| = |BQ| + |QC| = |BA|\,(\,|PB| + |PC|\,)\,|PA|^{-1}$

The three sides of the triangle are equal; therefore: $|PA| = |PB| + |PC|$

7.5 Let us firstly calculate the homothety ratio k, which is the quotient of homologous sides and the similarity ratio of both triangles:

$$AB^{-1}\,A'B' = k$$

Secondly we calculate the centre of the homothety by isolation from:

$$OA' = OA\,k$$

$$OA' - OA = OA\,(\,1 - k\,)$$

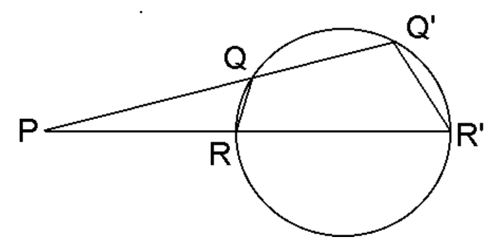
$$AA'\,(\,1 - k\,)^{-1} = OA$$

Since the segment AA' is known, this equation allows us to calculate the centre O :

$$O = A - AA'\,(\,1 - k\,)^{-1} = A - AA'\,(\,1 - AB^{-1}\,A'B'\,)^{-1}$$

7.6 Draw the line passing through P and the centre of the circle. This extended diameter cuts the circle in the points R and R'. See that the angles $R'RQ$ and $QQ'R'$ are supplementary because they intercept opposite arcs of the circle. Then the angles PRQ and $R'Q'P$ are equal. Therefore the triangle QRP and $R'Q'P$ are oppositely similar and we have:

Figure 16.2



$$PR^{-1}\,PQ = PR'\,PQ'^{-1} \;\Rightarrow\; PQ\,PQ' = PR\,PR'$$

Since PR and PR' are determined by P and the circle, the product $PQ\,PQ'$ (the *power* of P) is constant independently of the line PQQ'.

7.7 The bisector d of the angle ab divides the triangle abc in two triangles, which are obviously not similar! However we may apply the law of sines to both triangles to find:

$$\frac{|m|}{\sin ad} = \frac{|a|}{\sin dm} \qquad\qquad \frac{|n|}{\sin db} = \frac{|b|}{\sin nd}$$

The angles *nd* and *dm* are supplementary and sin *nd* = sin *dm*. On the other hand, the angles *ad* and *db* are equal because of the angle bisector. Therefore it follows that:

$$\frac{|m|}{|a|} = \frac{|n|}{|b|}$$

8. Properties of the triangles

8.1 Let A, B and C be the vertices of the given triangle with anticlockwise position. Let S, T and U be the vertices of the three equilateral triangles drawn over the sides AB, BC and CA respectively. Let P, Q and R be the centres of the triangles ABS, BCT and CAU respectively. The side CU is obtained from AC through a rotation of $2\pi/3$:

$$CU = AC\, t \qquad \text{with} \qquad t = 1_{2\pi/3} = \cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}$$

CR is 2/3 of the altitude of the equilateral triangles ACU ; therefore is 1/3 of the diagonal of the parallelogram formed by CA and CU:

$$CR = \frac{CA + CU}{3} = \frac{CA(1-t)}{3}$$

The same argument applies to the other equilateral triangles:

$$BQ = \frac{BC(1-t)}{3} \qquad\qquad AP = \frac{AB(1-t)}{3}$$

From where P, Q and R as functions of A, B and C are obtained:

$$P = A + \frac{AB(1-t)}{3} \qquad Q = B + \frac{BC(1-t)}{3} \qquad R = C + \frac{CA(1-t)}{3}$$

Let us calculate the vector PQ:

$$PQ = Q - P = B - A + \frac{(C - B - B + A)(1-t)}{3}$$

Introducing the centroid: $\quad G = \dfrac{A+B+C}{3}$

$$PQ = AB + BG\,(1-t) = AG - BG\,t$$

Analogously:

$$QR = BG - CG\,t \qquad\qquad RP = CG - AG\,z$$

Now we apply a rotation of $2\pi/3$ to PQ in order to obtain QR:

$$PQ\,t = (\,AG - BG\,t\,)\,t = AG\,t - BG\,t^2$$

Since t is a third root of the unity ($1 + t + t^2 = 0$) then:

$$PQ\,t = AG\,t + BG + BG\,t = AG\,t + BG\,t + CG\,t + BG - CG\,t$$

$$= 3\,GG + BG - CG\,t = BG - CG\,t = QR$$

Through the same way we find:

$$QR\,t = RP \qquad\qquad RP\,t = PQ$$

Therefore P, Q and R form an equilateral triangle with centre in G, the centroid of the triangle ABC:

$$\frac{P+Q+R}{3} = \frac{A+B+C}{3} + \frac{AB+BC+CA}{3}(1-t) = G$$

8.2 The substitution of the centroid G of the triangle ABC gives:

$$3\,PG^2 = 3\left[P - \frac{A+B+C}{3}\right]^2 = 3\,P^2 - 2\,P\cdot(A+B+C) + \frac{(A+B+C)^2}{3}$$

$$= 3\,P^2 - 2\,P\cdot(A+B+C) + \frac{A^2+B^2+C^2+2\,A\cdot B+2\,B\cdot C+2\,C\cdot A}{3}$$

$$= \left[(A-P)^2+(B-P)^2+(C-P)^2\right] + \frac{2\left(-A^2-B^2-C^2+A\cdot B+B\cdot C+C\cdot A\right)}{3}$$

$$= PA^2+PB^2+PC^2 - \frac{(B-A)^2+(C-B)^2+(A-C)^2}{3}$$

$$= PA^2+PB^2+PC^2 - \frac{AB^2+BC^2+CA^2}{3}$$

8.3 a) Let us calculate the area of the triangle GBC:

$$GB \wedge BC = [\,-a\,A + (\,1-b\,)\,B - c\,C\,] \wedge BC = [\,-a\,A + (\,a+c\,)\,B - c\,C\,] \wedge BC =$$

$$= (\,a\,AB + c\,CB\,) \wedge BC = a\,AB \wedge BC \qquad \Rightarrow \qquad a = \frac{GB \wedge BC}{AB \wedge BC}$$

The proof is analogous for the triangles GCA and GAB.

b) Let us develop PG^2 following the same way as in the exercise 8.2:

$$PG^2 = [P - (a\,A + b\,B + c\,C)\,]^2 = P^2 - 2\,P\cdot(a\,A + b\,B + c\,C) + (a\,A + b\,B + c\,C)^2$$

$$= (a + b + c)\,P^2 - 2\,a\,P\cdot A - 2\,b\,P\cdot B - 2\,c\,P\cdot C + a^2 A^2 + b^2 B^2 + c^2 C^2 +$$

$$+ 2\,a\,b\,A\cdot B + 2\,b\,c\,B\cdot C + 2\,c\,a\,C\cdot A$$

$$= a\,(A - P)^2 + b\,(B - P)^2 + c\,(C - P)^2 + a\,(a - 1)\,A^2 + b\,(b - 1)\,B^2$$

$$+ c\,(c - 1)\,C^2 + 2\,a\,b\,A\cdot B + 2\,b\,c\,B\cdot C + 2\,c\,a\,C\cdot A$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,(b + c)\,A^2 - b\,(c + a)\,B^2 - c\,(a + b)\,C^2$$

$$+ 2\,a\,b\,A\cdot B + 2\,b\,c\,B\cdot C + 2\,c\,a\,C\cdot A$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,b\,(B - A)^2 - b\,c\,(C - B)^2 - c\,a\,(A - C)^2$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,b\,AB^2 - b\,c\,BC^2 - c\,a\,CA^2$$

The Leibniz's theorem is a particular case of the Apollonius' lost theorem for $a = b = c = 1/3$.

8.4 Let us consider the vertices A, B, C and D ordered clockwise on the perimeter. Since AP is AB turned $\pi/3$, BQ is BC turned $\pi/3$, etc, we have:

$$z = \cos\pi/3 + e_{12}\sin\pi/3$$

$$AP = AB\,z \qquad BQ = BC\,z \qquad CR = CD\,z \qquad DS = DA\,z$$

$$PR\cdot QS = (PA + AC + CR)\cdot(QB + BD + DS)$$

$$= [\,(CD - AB)\,z + AC\,]\cdot[\,(DA - BC)\,z + BD\,]$$

$$= [\,(-AC + BD)\,z + AC\,]\cdot[\,(-AC - BD)\,z + BD\,] =$$

$$= [\,(-AC + BD)\,z\,]\cdot[\,(-AC - BD)\,z\,] + [\,(-AC + BD)\,z\,]\cdot BD$$

$$+ AC\cdot[\,(-AC - BD)\,z + BD\,]$$

The inner product of two vectors turned the same angle is equal to that of these vectors before the rotation. We use this fact for the first product. Also we must develop the other products in geometric products and permute vectors and the complex number z:

$$PR\cdot QS = (-AC + BD)\cdot(-AC - BD) + (-AC\,BD - BD\,AC - AC^2 + BD^2)\,\frac{z + z^*}{2}$$

$$+ AC\cdot BD = \frac{AC^2 - BD^2}{2}$$

Therefore $|AC| = |BD| \Leftrightarrow PR\cdot QS = 0$. The statement b) is proved through an analogous way.

8.5 The median is the segment going from a vertex to the midpoint of the opposite side:

$$m_A^2 = \left(\frac{B+C}{2} - A\right)^2 = \left(\frac{AB}{2} + \frac{AC}{2}\right)^2 = \frac{AB^2}{4} + \frac{AC^2}{4} + \frac{AB \cdot AC}{2}$$

$$= \frac{AB^2}{2} + \frac{AC^2}{2} + \frac{2\, AB \cdot AC - AB^2 - AC^2}{4} = \frac{AB^2}{2} + \frac{AC^2}{2} - \frac{BC^2}{4}$$

8.6 If E is the intersection point of the bisector of B with the line parallel to the bisector of A, then the following equality holds:

$$E = B + b\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = C + a\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right)$$

where a, b are real. Arranging terms we obtain a vectorial equality:

$$-a\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) + b\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = BC$$

The linear decomposition yields after simplification:

$$a = -\frac{|BC|\,|CA|}{|AB| + |BC| + |CA|}$$

In the same way, if D is the intersection of the bisector of A with the line parallel to the bisector of B we have:

$$D = A + c\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) = C + d\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right)$$

where c and d are real. Arranging terms:

$$-c\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) + d\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = CA$$

After simplification we obtain:

$$d = -\frac{|BC|\,|CA|}{|AB| + |BC| + |CA|}$$

Now we calculate the vector ED:

$$ED = D - E = \left(\frac{2\,AB}{|\,AB\,|} - \frac{BC}{|\,BC\,|} + \frac{AC}{|\,AC\,|} \right) \frac{|\,BC\,|\;|\,CA\,|}{|\,AB\,| + |\,BC\,| + |\,CA\,|}$$

Then the direction of the line ED is given by the vector v:

$$v = \frac{2\,AB}{|\,AB\,|} - \frac{BC}{|\,BC\,|} + \frac{AC}{|\,AC\,|} = AB \left(\frac{2}{|\,AB\,|} + \frac{1}{|\,AC\,|} \right) + BC \left(-\frac{1}{|\,BC\,|} + \frac{1}{|\,AC\,|} \right)$$

When the vector v has the direction AB, the second summand vanishes, $|AC| = |BC|$ and the triangle becomes isosceles.

8.7 Let us indicate the sides of the triangle ABC with a, b and c in the following form:

$$a = BC \qquad b = CA \qquad c = AB$$

Suppose without loss of generality that P lies on the side BC and Q on the side AC . Hence:

$$P = k\,B + (1 - k)\,C \qquad\qquad Q = l\,A + (1 - l)\,C$$

$$CP = k\,CB = -\,k\,a \qquad\qquad CQ = l\,CA = l\,b$$

where k and l are real and $0 < k, l < 1$. Now the segment PQ is obtained:

$$PQ = k\,a + l\,b$$

Since the area of the triangle CPQ must be the half of the area of the triangle ABC, it follows that:

$$CP \wedge CQ = \frac{1}{2} CB \wedge CA \qquad \Rightarrow \qquad (-\,k\,a) \wedge (l\,b) = -\frac{1}{2}\,a \wedge b \qquad \Rightarrow \qquad k\,l = \frac{1}{2}$$

The substitution into PQ gives:

$$PQ = k\,a + \frac{1}{2k} b$$

a) If u is the vector of the given direction, PQ is perpendicular when $PQ \cdot u = 0$ which results in:

$$\left(k\,a + \frac{1}{2k} b \right) \cdot u = 0$$

whence one obtains:

$$k = \sqrt{-\frac{b \cdot u}{2\,a \cdot u}} \qquad\qquad PQ = a\sqrt{-\frac{b \cdot u}{2\,a \cdot u}} + b\sqrt{-\frac{a \cdot u}{2\,b \cdot u}}$$

In this case the solutions only exist when $a \cdot u$ and $b \cdot u$ have different signs. However we can also choose the point P (or Q) lying on the another side c, what gives an analogous solution containing the inner product $c \cdot u$. Note that if $a \cdot u$ and $b \cdot u$ have the same sign, then $c \cdot u$ have the opposite sign because:

$$a + b + c = 0 \qquad \Rightarrow \qquad c \cdot u = -\, a \cdot u - b \cdot u$$

what warrants there is always a solution.

b) Let us calculate the square of PQ:

$$PQ^2 = k^2 a^2 + \frac{1}{4k^2} b^2 + a \cdot b$$

By equating the derivative to zero, one obtains the value of k for which PQ^2 is minimum:

$$k = \sqrt{\frac{|\,b\,|}{2\,|\,a\,|}}$$

Then we obtain the segment PQ and its length:

$$PQ = a\sqrt{\frac{|\,b\,|}{2\,|\,a\,|}} + b\sqrt{\frac{|\,a\,|}{2\,|\,b\,|}} \qquad\qquad |\,PQ\,| = \sqrt{|\,a\,|\,|\,b\,| + a \cdot b}$$

c) Every point may be written as linear combination of the three vertices of the triangle the sum of the coefficients being equal to the unity:

$$R = x\, A + y\, B + (1 - x - y)\, C \qquad \Leftrightarrow \qquad CR = x\, CA + y\, CB$$

where all the coefficients are comprised between 0 and 1:

$$0 < x,\, y,\, 1 - x - y < 1 \qquad x = \frac{CR \wedge CB}{CA \wedge CB} \qquad y = \frac{CA \wedge CR}{CA \wedge CB}$$

Now the point R must lie on the segment PQ, that is, P, Q and R must be aligned. Then the determinant of their coordinates will vanish:

$$\det(P,Q,R) = \begin{vmatrix} 0 & k & 1-k \\ l & 0 & 1-l \\ x & y & 1-x-y \end{vmatrix} = \begin{vmatrix} 0 & k & 1-k \\ \dfrac{1}{2\,k} & 0 & 1-\dfrac{1}{2\,k} \\ x & y & 1-x-y \end{vmatrix} = 0$$

$$2\,x\,k^2 - k + y = 0 \qquad \Rightarrow \qquad k = \frac{1 \pm \sqrt{1 - 8\,x\,y}}{4\,x} \qquad \text{and} \qquad l = \frac{1}{2\,k} = \frac{1 \mp \sqrt{1 - 8\,x\,y}}{4\,y}$$

There is only solution for a positive discriminant:

$$1 > 8\,x\,y$$

The limiting curve is an equilateral hyperbola on the plane *x*-*y*. Since the triangle may be obtained through an affinity, the limiting curve for the triangle is also a hyperbola although not equilateral.

If the extremes *P* and *Q* could move along the prolongations of the sides of the triangle without limitations, this would be the unique condition. However in this problem the point *P* must lie between *C* and *B*, and *Q* must lie between *C* and *A*. It means the additional condition:

$$\frac{1}{2} < k < 1 \qquad \Leftrightarrow \qquad \frac{1}{2} < l < 1 \qquad \Leftrightarrow \qquad \frac{1}{2} < \frac{1 \pm \sqrt{1 - 8\,x\,y}}{4\,x} < 1$$

From the first solution (root with sign +) we have:

$$2\,x - 1 < \sqrt{1 - 8\,x\,y} < 4\,x - 1$$

only existing for $x > 1/4$. For $1/4 < x < 1/2$ the left hand side is negative so that the unique restriction is the right hand side. By squaring it we obtain:

$$1 - 8\,x\,y < (4\,x - 1)^2 \qquad \Rightarrow \qquad 0 < 2\,x + y - 1 \qquad \text{for} \qquad \frac{1}{4} < x < \frac{1}{2}$$

For $x > 1/2$ the right hand side is higher than 1 so that the unique restriction is the left hand side:

$$(2\,x - 1)^2 < 1 - 8\,x\,y \qquad \Rightarrow \qquad x + 2\,y - 1 < 0 \qquad \text{for} \qquad x > \frac{1}{2}$$

For the second solution (root with sign –) we have:

$$2\,x - 1 < -\sqrt{1 - 8\,x\,y} < 4\,x - 1$$

only existing for $x < 1/2$. For $1/4 < x < 1/2$ the right hand side is positive so that the unique restriction is the left hand side:

$$(2\,x - 1)^2 > 1 - 8\,x\,y \qquad \Rightarrow \qquad x + 2\,y - 1 > 0 \qquad \text{for} \qquad \frac{1}{4} < x < \frac{1}{2}$$

For $x < 1/4$ both members are negative and after squaring we have:

$$(2x-1)^2 > 1-8xy > (4x-1)^2 \qquad \Rightarrow \qquad x+2y-1>0>2x+y-1 \quad \text{for } x<\frac{1}{4}$$

All these conditions are plotted in the figure 16.3.

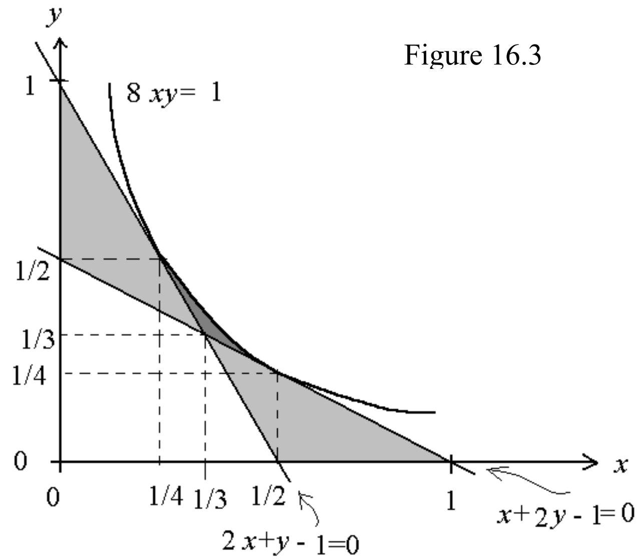


Figure 16.3

It is easily proved that the limiting lines touch the hyperbola at the points (1/4, 1/2) and (1/2, 1/4). For both triangles painted with light grey there is a unique solution. In the region painted with dark grey there are two solutions, that is, there are two segments passing through the point R and dividing the triangle in two parts with equal area. By means of an affinity the plot in the figure 16.3 is transformed into any triangle (figure 16.4). The affinity is a linear transformation of the coordinates, which converts the limiting lines into the medians of the triangle intersecting at the centroid and the hyperbola into another non equilateral hyperbola:

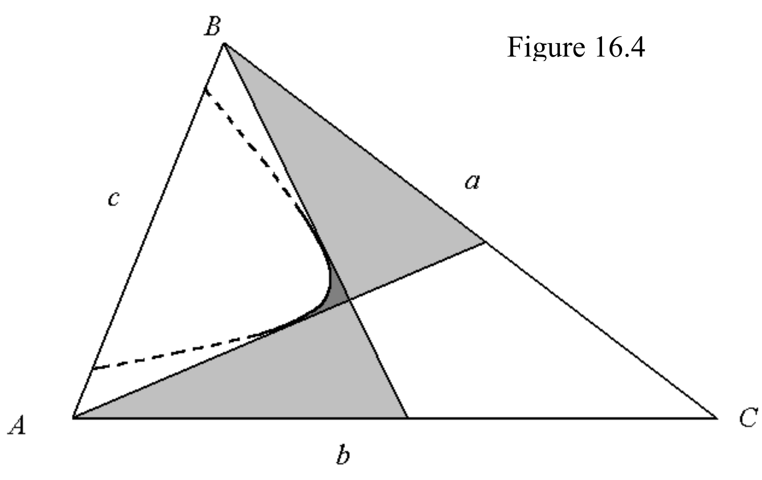


Figure 16.4

Observe that the medians divide the triangle ABC in six triangles. If a point R lies outside the shadowed triangles, we can always support the extremes of the segment PQ passing through R in another pair of sides. This means that the division of the triangle is always possible provided of the suitable choice of the sides. On the other hand, if we consider all the possibilities, the points neighbouring the centroid admit three segments.

9. Circles

9.1 $(A-M)^2+(B-M)^2+(C-M)^2=k$

$$3M^2-2M\cdot(A+B+C)+A^2+B^2+C^2=k$$

Introducing the centroid $G=(A+B+C)/3$:

$$M^2-2M\cdot G+G^2=\frac{k}{3}-\frac{2\left(A^2+B^2+C^2+A\cdot B+B\cdot C+C\cdot A\right)}{9}$$

$$(M-G)^2=\frac{k}{3}-\frac{(B-A)^2+(C-B)^2+(A-C)^2}{9}$$

$$GM^2=\frac{k}{3}-\frac{AB^2+BC^2+CA^2}{9}=r^2$$

M runs on a circumference with radius r centred at the centroid of the triangle when k is higher than the arithmetic mean of the squares of the three sides:

$$k\geq\frac{AB^2+BC^2+CA^2}{3}$$

9.2 Let A and B be the fixed points and P any point on the searched geometric locus. If the ratio of distances from P to A and B is constant it holds that:

$$|PA|=k\,|PB| \quad\Leftrightarrow\quad PA^2=k^2\,PB^2 \quad \text{with } k \text{ being a real positive number}$$

$$A^2-2A\cdot P+P^2=k^2(B^2-2B\cdot P+P^2)$$

$$(1-k^2)P^2-2(A-k^2B)\cdot P=k^2B^2-A^2$$

$$P^2-\frac{2P\cdot(A-k^2B)}{1-k^2}=\frac{k^2B^2-A^2}{1-k^2}$$

We indicate by O the expression: $O=\dfrac{A-k^2B}{1-k^2}$. Adding O^2 to both members we obtain:

$$OP^2=\frac{k^2B^2-A^2}{1-k^2}+\frac{\left(A-k^2B\right)^2}{\left(1-k^2\right)^2}$$

$$OP^2 = \frac{k^2\left(B^2 - 2\,A\cdot B + A^2\right)}{\left(1-k^2\right)^2} = \frac{k^2\,AB^2}{\left(1-k^2\right)^2}$$

Therefore, the points P form a circumference centred at O, which is a point of the line AB, with radius:

$$|\,OP\,| = \frac{k\,|\,AB\,|}{1-k^2}$$

9.3 Let us draw the circumference circumscribed to the triangle ABC (figure 16.5) and let D be any point on this circumference. Let P, Q and R be the orthogonal projections of P on the sides AB, BC and CA respectively. Note that the triangles ADP and CDR are similar because of $DAB = \pi - BCD = DCR$. The similarity of both triangles is written as:

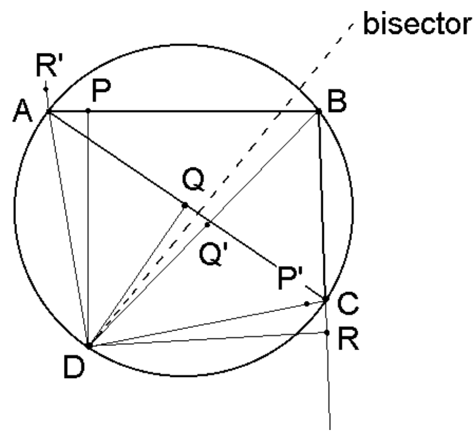


Figure 16.5

$$DC\,DA^{-1} = DR\,DP^{-1}$$

The triangles ADQ and BDR are also similar because of the equality of the inscribed angles $CAD = CBD$:

$$DB\,DA^{-1} = DR\,DQ^{-1}$$

Since the angles $RDC = PDA$ and $RDB = QDA$, the bisector of the angle BDQ is also bisector of the angle CDP and RDA, being its direction vector:

$$v = \frac{DA}{|\,DA\,|} + \frac{DR}{|\,DR\,|} = \frac{DB}{|\,DB\,|} + \frac{DQ}{|\,DQ\,|} = \frac{DC}{|\,DC\,|} + \frac{DP}{|\,DP\,|}$$

Let P', Q' and R' be the reflected points of P, Q and R with respect to the bisector:

$$DR' = v^{-1}\,DR\,v$$

Multiplying by DA we have:

$$DR'\,DA = v^{-1}\,DR\,v\,DA = |\,DR\,|\;|\,DA\,|$$

which is an expected result since the reflection of R with respect the bisector of the angle RDA yields always a point R' aligned with D and A. Analogously for B and C we find that the points D, B and Q' are aligned and also the points D, C and P':

$$DQ'\,DB = v^{-1}\,DQ\,v\,DR\,DQ^{-1}\,DA = v^{-1}\,DR\,v\,DA = |\,DR\,|\;|\,DA\,|$$

$$DP'\,DC = v^{-1}\,DP\,v\,DR\,DP^{-1}\,DA = v^{-1}\,DR\,v\,DA = |\,DR\,|\;\;|\,DA\,|$$

Now we see that the points R', Q' and P' are the transformed of A, B and C under an inversion with centre D:

$$DR'\,DA = DQ'\,DB = DP'\,DC$$

Since D lies on the circumference passing through A, B and C, the inversion transforms them into aligned points and the reflection preserves this alignment so that P, Q and R are aligned.

9.4 If $m = a + b$ and $n = b + c$ then:

$$m^2 n^2 = (a+b)^2 (b+c)^2 = \left(a^2 + a\,b + b\,a + b^2\right)\left(b^2 + b\,c + c\,b + c^2\right)$$

$$= a^2 b^2 + a^2 b\,c + a^2 c\,b + a^2 c^2 + a\,b^3 + a\,b^2 c + a\,b\,c\,b + a\,b\,c^2 +$$

$$+ b\,a\,b^2 + b\,a\,b\,c + b\,a\,c\,b + b\,a\,c^2 + b^4 + b^3 c + b^2 c\,b + b^2 c^2$$

Taking into account that $d = -(a + b + c)$ then:

$$d^2 = (a+b+c)^2 = a^2 + b^2 + c^2 + a\,b + b\,a + b\,c + c\,b + c\,a + a\,c$$

so we may rewrite the former equality:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 + a^2 b\,c + a^2 c\,b + a\,b\,c\,b + a\,b\,c^2 + b\,a\,b\,c + b\,a\,c^2$$

We apply the permutative property to some terms:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 + a^2 b\,c + a\,b\,c\,a + a\,b\,c\,b + a\,b\,c^2 + b\,a\,b\,c + c\,a\,b\,c$$

in order to arrive at a fully symmetric expression:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - d\,a\,b\,c - a\,b\,c\,d$$

Now we express the product of each pair of vectors using the exponential function of their angle:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - |\,a\,|\,|\,b\,|\,|\,c\,|\,|\,d\,|\left(\exp\left((2\pi - \gamma - \alpha)\,e_{12}\right) + \exp\left((2\pi - \beta - \delta)\,e_{12}\right)\right)$$

Adding the arguments of the exponential, simplifying and taking into account that $\alpha + \beta + \gamma + \delta = \pi$.

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - 2\,|\,a\,|\,|\,b\,|\,|\,c\,|\,|\,d\,|\cos(\alpha + \gamma)$$

9.5 Let A, B and C be the midpoints of the sides of any triangle PQR :

$$A = \frac{Q+R}{2} \qquad B = \frac{R+P}{2} \qquad C = \frac{P+Q}{2}$$

The centre of the circumferences RAB, PBC and QCA will be denoted as D, E and F respectively. The circumference PBC is obtained from the circumscribed circumference PQR by means of a homothety with centre P. Then if O is the circumcentre of the triangle PQR, the centre of the circumference PBC is located at half distance from R:

$$E = \frac{P+O}{2} \qquad F = \frac{Q+O}{2}$$

$$EA = \frac{PQ}{2} + \frac{OR}{2} \qquad FB = -\frac{PQ}{2} + \frac{OR}{2} \qquad EF = \frac{PQ}{2}$$

Let Z be the intersection of EA with FB. Then:

$$Z = a\,A + (1-a)E = b\,B + (1-b)F$$

Arranging terms we find:

$$a\,EA - b\,FB = EF$$

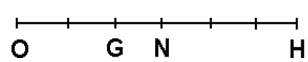
The linear decomposition gives:

$$a = \frac{EF \wedge FB}{EA \wedge FB} \qquad -b = \frac{EA \wedge EF}{EA \wedge FB}$$

$$Z = \frac{EF \wedge FB}{EA \wedge FB} A + \frac{FA \wedge FB}{EA \wedge FB} E = \frac{A}{2} + \frac{E}{2} = \frac{P+Q+R+O}{4}$$

which is invariant under cyclic permutation of the vertices P, Q and R. Then the lines EA, FB and DC intersect in a unique point Z. On the other hand, the point Z lies on the Euler's line (figure 9.8):

Figure 9.8



$$Z = \frac{3\,G+O}{4}$$

9.6 We must prove that the inversion changes the single ratio of three very close points by its conjugate value, because the orientation of the angles is changed. Let us consider an inversion with centre O and radius r and let the points A', B', C' be the transformed of A, B, C under this inversion. Then:

$$OA' = r^2\,OA^{-1} \qquad OB' = r^2\,OB^{-1} \qquad OC' = r^2\,OC^{-1}$$

$$A'B' = OB' - OA' = r^2\,(OB^{-1} - OA^{-1}) = r^2\,OA^{-1}\,(OA - OB)\,OB^{-1}$$

$$= - r^2\, OA^{-1}\, AB\, OB^{-1}$$

In the same way: $A'C' = - r^2\, OA^{-1}\, AC\, OC^{-1}$
Let us calculate the single ratio of the transformed points:

$$(\, A',\, B',\, C'\,) = A'B'\, A'C'^{-1} = OA^{-1}\, AB\, OB^{-1}\, OC\, AC^{-1}\, OA =$$

$$= OB^{-1}\, AB\, AC^{-1}\, OC = (\, OA + AB\,)^{-1}\, AB\, AC^{-1}\, (\, OA + AC\,)$$

When B and C come near A, AB and AC tend to zero and the single ratio for the inverse points becomes the conjugate value of that for the initial points:

$$\lim_{B,C \to A} (A', B', C') = OA^{-1}\, AB\; AC^{-1}\, OA = AC^{-1} AB = (A, B, C)^*$$

10. Cross ratios and related transformations

10.1 Let $ABCD$ be a quadrilateral inscribed in a circle. We must prove the following equality:

$$|BC|\;|AD| + |AB|\;|CD| = |BD|\;|AC|$$

which is equivalent to:

$$\frac{|BC||AD|}{|BD||AC|} + \frac{|AB||CD|}{|BD||AC|} = 1$$

Now we identify these quotients with cross ratios of four points A, B, C and D on a circle:

$$(\, B\; A\; C\; D\,) + (\, B\; C\; A\; D\,) = 1$$

equality that always holds because if $(\, A\; B\; C\; D\,) = r$ then:

$$\frac{1}{r} + \frac{r-1}{r} = 1$$

When the points do not lie on a circle, the cross ratio is not the quotient of moduli but a complex number:

$$\frac{|BC||AD|}{|BD||AC|} \exp\left[e_{12}\left(CBD - CAD\right)\right] + \frac{|AB||CD|}{|BD||AC|} \exp\left[e_{12}\left(ABD - ACD\right)\right] = 1$$

According to the triangular inequality, the modulus of the sum of two complex numbers is lower than or equal to the sum of both moduli:

$$\frac{|BC||AD|}{|BD||AC|}+\frac{|AB||CD|}{|BD||AC|}\geq 1 \quad \Rightarrow \quad |BC||AD|+|AB||CD|\geq |BD||AC|$$

10.2 If A, B, C and D form a harmonic range their cross ratio is 2:

$$AC\,AD^{-1}\,BD\,BC^{-1}=2$$

In order to isolate D we must write BD as addition of BA and AD:

$$AC\,AD^{-1}\,(\,BA+AD\,)\,BC^{-1}=2$$

$$AC\,AD^{-1}\,BA\,BC^{-1}=2-AC\,BC^{-1}$$

$$AD^{-1}=AC^{-1}\,(\,2-AC\,BC^{-1}\,)\,BC\,BA^{-1}=(\,2\,AC^{-1}\,BC-1\,)\,BA^{-1}$$

$$AD=AB\,(\,1-2\,AC^{-1}\,BC\,)^{-1}$$

10.3 The homography preserves the cross ratio:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}=AC\,AD^{-1}\,BD\,BC^{-1}$$

which may be rewritten as:

$$A'C'\,A'D'^{-1}\,(\,B'A'+A'D'\,)\,(\,B'A'+A'C'\,)^{-1}=AC\,AD^{-1}\,(\,BA+AD\,)\,(\,BA+AC\,)^{-1}$$

When C and D approach to A, the vectors AC, AD, $A'C'$ and $A'D'$ tend to zero. So the single ratio of three very close points remains constant:

$$\lim_{C,D\to A} A'C'\;A'D'^{-1}=\lim_{C,D\to A} AC\;AD^{-1}$$

and therefore the angle between tangent vectors of curves. So the homography is a directly conformal transformation.

10.4 If F is a point on the homology axis we must prove that:

$$\{\,F, A\,B\,C\,D\,\}=\{\,F, A'\,B'\,C'\,D'\,\}$$

As we have seen, a point A' homologous of A is obtained through the equation:

$$OA'=OA\,[\,1-r\,v\wedge FA\,(\,v\wedge FO\,)^{-1}\,]^{-1}$$

where O is the centre of homology, F a point on the axis, v the direction vector of the axis and r the homology ratio. Then the vector FA' is:

$$FA'=FO+OA'=FO+OA\,[\,1-r\,v\wedge FA\,(\,v\wedge FO\,)^{-1}\,]^{-1}$$

$$= [\, FO\,(\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,) + OA\,]\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$= [\, FA - r\, FO\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

Analogously:

$$FB' = [\, FB - r\, FO\, v \wedge FB\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FB\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$FC' = [\, FC - r\, FO\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$FD' = [\, FD - r\, FO\, v \wedge FD\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FD\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

The product of two outer products is a real number so that we must only pay attention to the order of the first loose vectors FA and FO, etc:

$$FA' \wedge FC' = [\, FA \wedge FC - r\,(\, FA \wedge FO\, v \wedge FC + FO \wedge FC\, v \wedge FA\,)\,(\, v \wedge FO\,)^{-1}\,]$$
$$[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

Using the identity of the exercise 1.4, we have:

$$FA' \wedge FC' = [\, FA \wedge FC - r\, FA \wedge FC\, v \wedge FO\,(\, v \wedge FO\,)^{-1}\,]$$
$$[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$
$$= FA \wedge FC\,(\, 1 - r\,)\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

In the projective cross ratio all the factors except the first outer product are simplified:

$$\frac{FA' \wedge FC'\; FB' \wedge FD'}{FA' \wedge FD'\; FB' \wedge FC'} = \frac{FA \wedge FC\; FB \wedge FD}{FA \wedge FD\; FB \wedge FC}$$

10.5 a) Let us prove that the special conformal transformation is additive:

$$OP' = (\, OP^{-1} + v\,)^{-1} \quad \Rightarrow \quad OP'^{-1} = OP^{-1} + v$$

$$OP'' = (\, OP'^{-1} + w\,)^{-1} = (\, OP^{-1} + v + w\,)^{-1}$$

b) Let us extract the factor OP from OP':

$$OP'^{-1} = (\, 1 + v\, OP\,)\, OP^{-1} \quad \Rightarrow \quad OP' = OP\,(\, 1 + v\, OP\,)^{-1}$$

Then we have:

$$OA' = OA\,(\, 1 + v\, OA\,)^{-1} \qquad OB' = OB\,(\, 1 + v\, OB\,)^{-1}$$

$$OC' = OC\,(\, 1 + v\, OC\,)^{-1} \qquad OD' = OD\,(\, 1 + v\, OD\,)^{-1}$$

$$A'C' = OC' - OA' = OC\,(\,1 + v\,OC\,)^{-1} - OA\,(\,1 + v\,OA\,)^{-1} =$$

$$= [\,OC\,(\,1 + v\,OA\,) - OA\,(\,1 + v\,OC\,)\,]\,(\,1 + v\,OC\,)^{-1}\,(\,1 + v\,OA\,)^{-1}$$

$$= AC\,(\,1 + v\,OC\,)^{-1}\,(\,1 + v\,OA\,)^{-1}$$

Analogously:

$$B'D' = BD\,(\,1 + v\,OB\,)^{-1}\,(\,1 + v\,OD\,)^{-1}$$

$$A'D' = AD\,(\,1 + v\,OA\,)^{-1}\,(\,1 + v\,OD\,)^{-1}$$

$$B'C' = BC\,(\,1 + v\,OB\,)^{-1}\,(\,1 + v\,OC\,)^{-1}$$

From where it follows that the complex cross ratio is preserved and it is a special case of homography:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = AC\,(\,1 + v\,OC\,)^{-1}\,(\,1 + v\,OD\,)\,AD^{-1}$$

$$BD\,(\,1 + v\,OD\,)^{-1}\,(\,1 + v\,OC\,)^{-1}\,BC^{-1} = AC\,AD^{-1}\,BD\,BC^{-1}$$

10.6 If the points A, B and C are invariant under a certain homography, then for any other point D' the following equality is fulfilled:

$$(\,A\,B\,C\,D\,) = (\,A\,B\,C\,D'\,)$$

$$AC\,AD^{-1}\,BD\,BC^{-1} = AC\,AD'^{-1}\,BD'\,BC^{-1}$$

The simplification of factors yields:

$$AD^{-1}\,BD = AD'^{-1}\,BD' \quad\Rightarrow\quad AD^{-1}\,(\,BA + AD\,) = AD'^{-1}\,(\,BA + AD'\,) \quad\Rightarrow$$

$$\Rightarrow\quad AD^{-1}\,BA = AD'^{-1}\,BA \quad\Rightarrow\quad AD^{-1} = AD'^{-1} \quad\Rightarrow\quad D = D'$$

10.7. **a)** From the definition of antigraphy we have:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = (\,AC\,AD^{-1}\,BD\,BC^{-1}\,)^* = BC^{-1}\,BD\,AD^{-1}\,AC$$

If C and D approach A, then BC, $BD \rightarrow BA$:

$$\lim_{C',D'\to A} A'C'\,A'D'^{-1} = \lim_{C,D\to A} AD^{-1}AC$$

So the antigraphy is an opposite conformal transformation.

b) The composition of two antigraphies is always a homography because twice the conjugation of a complex number is the identity, so that the complex cross ratio is preserved.

c) The inversion conjugates the cross ratio. Consequently an odd number of inversions is always an antigraphy.

d) If an antigraphy has three invariant points, then all the points lying on the circumference passing through these points are invariant. Let us prove this statement: if *A*, *B* and *C* are the invariant points and *D* belongs to the circumference passing through these points, then the cross ratio is real:

$$(\,A\;B\;C\;D\,) = (\,A\;B\;C\;D\,)^* = (\,A\;B\;C\;D'\,)$$

from where $D = D'$ as proved in the exercise 10.6. If *D* does not belong to this circumference, $D' \neq D$ and the cross ratio ($A\;B\;C\;D$) becomes conjugate. A geometric transformation that preserves a circumference and conjugates the cross ratio can only be an inversion because there is a unique point D' for each point *D* fulfilling this condition. The centre and radius of inversion are those for the circumference *ABC*.

10.8 Consider the projectivity:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

If A', B' and C' are the images of *A, B* and *C* then the coefficients a', b', c' must be proportional to *a*, *b*, *c*:

$$a' = \frac{k\,a}{k\,a + l\,b + m\,c} \qquad b' = \frac{l\,b}{k\,a + l\,b + m\,c} \qquad c' = \frac{m\,c}{k\,a + l\,b + m\,c}$$

where the values are already normalised. Three points are collinear if the determinant of the barycentric coordinates is zero, what only happens if the initial points are also collinear:

$$\begin{vmatrix} a'_D & b'_D & c'_D \\ a'_E & a'_E & a'_E \\ a'_F & a'_F & a'_F \end{vmatrix} = \frac{k\;l\;m \begin{vmatrix} a_D & b_D & c_D \\ a_E & a_E & a_E \\ a_F & a_F & a_F \end{vmatrix}}{\left(k\;a_D + l\;b_D + m\;c_D\right)\left(k\;a_E + l\;b_E + m\;c_E\right)\left(k\;a_F + l\;b_F + m\;c_F\right)} = 0$$

10.9 In the dual plane the six sides of the hexagon are six points. Since the sides *AB*, *CD* and *EF* (or their prolongations) pass through *P* and *BC*, *DE* and *FA* pass through *Q*, this means that the dual points are alternatively aligned in two dual lines *P* and *Q*. By the Pappus' theorem the dual points *AD*, *BE* and *CF* lie on a dual line *X*, that is, these lines joining opposite vertices of the hexagon are concurrent in the point *X*. Moreover, as proved for the Pappus' theorem the cross ratios fulfil the equalities in the left hand side:

$$(BE\;AD\;FC\;P) = (PQ\;AB\;FE\;CD) \qquad \Leftrightarrow \qquad \{X, B\;A\;F\;P\} = \{P, Q\;A\;F\;C\}$$

$$(BE\;AD\;FC\;Q) = (PQ\;DE\;CB\;FA) \qquad \Leftrightarrow \qquad \{X, B\;A\;F\;Q\} = \{Q, P\;D\;C\;F\}$$

Since the cross ratio of a pencil of lines is equal to the cross ratio of the dual points, the equalities in the right hand side follow immediately.

11. Conics

11.1 The starting point is the central equation of a conic:

$OA = OQ \cos \alpha + OR \sin \alpha \qquad OB = OQ \cos \beta + OR \sin \beta$

$OC = OQ \sin \gamma + OR \sin \gamma \qquad OD = OQ \cos \delta + OR \sin \delta$

$OX = OQ \cos \chi + OR \sin \chi$

$XA = OA - OX = OQ (\cos \alpha - \cos \chi) + OR (\sin \alpha - \sin \chi)$

$XC = OC - OA = OQ (\cos \gamma - \cos \chi) + OR (\sin \gamma - \sin \chi)$

$$\begin{aligned} XA \wedge XC &= OQ \wedge OR \, [(\cos \gamma - \cos \chi) (\sin \alpha - \sin \chi) - (\cos \alpha - \cos \chi) (\sin \gamma - \sin \chi)] \\ &= OQ \wedge OR \, [\cos \gamma \sin \alpha - \cos \gamma \sin \chi - \cos \chi \sin \alpha - \cos \alpha \sin \gamma \\ &\qquad + \cos \alpha \sin \chi + \cos \chi \sin \gamma] \\ &= OQ \wedge OR \, [\sin(\alpha - \gamma) + \sin(\gamma - \chi) + \sin(\chi - \alpha)] \\ &= -4 \, OQ \wedge OR \sin \frac{\alpha - \gamma}{2} \sin \frac{\gamma - \chi}{2} \sin \frac{\chi - \alpha}{2} \end{aligned}$$

From where it follows that:

$$\frac{XA \wedge XC \; XB \wedge XD}{XA \wedge XD \; XB \wedge XC} = \frac{\sin \frac{\alpha - \gamma}{2} \sin \frac{\beta - \delta}{2}}{\sin \frac{\alpha - \delta}{2} \sin \frac{\beta - \gamma}{2}}$$

11.2 The central equation of an ellipse of centre *O* with semiaxis *OQ* and *OR* is:

$OP = OQ \cos \theta + OR \sin \theta \quad \Rightarrow \quad P = O (1 - \cos\theta - \sin \theta) + Q \cos \theta + R \sin \theta$

There is always an affinity transforming an ellipse into a circle:

$O' = (0, 0) \qquad Q' = (1, 0) \qquad R' = (0, 1)$

$P' = O' (1 - \cos \theta - \sin \theta) + Q' \cos \theta + R' \sin \theta = (\cos \theta , \sin \theta)$

Let us now consider any point *E* and a circle. A line passing through *E* cuts the circle in the points *A* and *B*; another line passing also through *E* but with different direction cuts the ellipse in the points *C* and *D*. Since the power of a point *E* with respect to the circle is constant we have:

$$EA\, EB = EC\, ED \qquad\qquad EA\, EB^{-1}\; |\, EB\,|^{2} = EC\, ED^{-1}\,|\, ED\,|^{2}$$

The affinity preserves the single ratio of three aligned points:

$$EA\, EB^{-1} = E'A'\, E'B'^{-1} \qquad\qquad EC\, ED^{-1} = E'C'\, ED'^{-1}$$

Then:

$$E'A'\, E'B'^{-1}\,|\, EB\,|^{2} = E'C'\, E'D'^{-1}\,|\, ED\,|^{2}$$

$$\frac{E'A'\; E'B'}{E'C'\; E'D'} = \frac{|\, E'B'\,|^{2}\;|\, ED\,|^{2}}{|\, EB\,|^{2}\;|\, E'D'\,|^{2}}$$

and this quotient is constant for two given directions because the preservation of the single ratio implies that the distances in each direction are enlarged by a constant ratio:

$$\frac{|\, EA\,|}{|\, E'A'\,|} = \frac{|\, EB\,|}{|\, E'B'\,|}$$

11.3 If a diameter is formed by the midpoints of the chords parallel to the conjugate diameter, then it is obvious that the tangent to the point where the length of the chord vanishes is also parallel to the conjugate diameter. This evidence may be proved by derivation of the central equation:

$$OP = OQ \cos\,\theta\; + OR \sin\,\theta$$

$$\frac{d\,OP}{d\theta} = -\,OQ\,\sin\theta + OR\cos\theta$$

That is, OP and $dOP\,/\,d\theta$ (having the direction of the tangent to P) are conjugate radius. The area a of the parallelogram circumscribed to the ellipse is the outer product of both conjugate diameters, which is independent of θ:

$$a = 4\;(OQ \cos\theta\; + OR \sin\theta) \wedge (\,-OQ \sin\theta\; + OR \cos\theta) = 4\; OQ \wedge OR$$

11.4 The area of a parallelogram formed by two conjugate diameters of the hyperbola is also independent of ψ:

$$a = 4\;(OQ \cosh\,\psi\; + OR \sinh\,\psi) \wedge (\; OQ \sinh\,\psi\; + OR \cosh\,\psi) = 4\; OQ \wedge OR$$

11.5 Let us prove the statement for a circle. The intersections R and R' of a circumference with a line passing through P are given by (fig. 9.3):

$$|\, PR\,| = |\, PF\,|\cos\alpha - \sqrt{r^{2} - PF^{2}\sin^{2}\alpha}$$

$$|PR'| = |PF| \cos\alpha + \sqrt{r^2 - PF^2 \sin^2\alpha}$$

The point *M* located between *R* and *R′* forming harmonic range with *P*, *R* and *R′* fulfils the condition:

$$(P\,R\,M\,R') = \frac{|PM||RR'|}{|PR'||RM|} = 2$$

Then: $$|PM| = \frac{2|PR||PR'|}{|PR| + |PR'|}$$

The numerator is the power of *P* (constant for any line passing through *P*) and after operations we obtain the polar equation of the geometric locus of the points *M*:

$$|PM| = \frac{PT^2}{|PF| \cos\alpha}$$

which is the polar line. When α is the angle *TPF* (figure 9.2) we have $|PM| = |PT|$, that is, the polar passes through the points *T* of tangency to the circumference.

Now, by means of any projectivity (e.g. a homology) the circle is transformed into any conic. Since it preserves the projective cross ratio, the points *M* also form an aligned harmonic range in the conic, so the polar also passes through the touching points of the tangents drawn from *P*.

11.6 Let the points be denoted by:

$$A = (1,1) \qquad B = (2,3) \qquad C = (1,4) \qquad D = (0,2) \qquad E = (2,5)$$

Then: $EA \wedge EC = -3 \qquad EB \wedge ED = -4 \qquad EA \wedge ED = -5 \qquad EB \wedge EC = -2$

The cross ratio is:

$$\{E, A\,B\,C\,D\} = \frac{EA \wedge EC\ EB \wedge ED}{EA \wedge ED\ EB \wedge EC} = \frac{6}{5}$$

Now a generic point of the conic will be $X = (x, y)$:

$$XA \wedge XC = -3\,x + 3 \qquad XB \wedge XD = x - 2\,y + 4$$

$$XA \wedge XD = -x - y + 2 \qquad XB \wedge XC = -x - y + 5$$

Applying the Chasles' theorem we find the equation of the conic:

$$\frac{XA \wedge XC \; XB \; XD}{XA \wedge XD \; XB \wedge XC} = \frac{(-3x+3)(x-2y+4)}{(-x-y+2)(-x-y+5)} = \frac{6}{5}$$

After simplifying we arrive at the point equation:

$$7x^2 + 2y^2 - 6xy + x - 4y = 0$$

which may be written in the form:

$$\begin{pmatrix} 1-x-y & x & y \end{pmatrix} \begin{pmatrix} 0 & 1/2 & -2 \\ 1/2 & 8 & -9/2 \\ -2 & -9/2 & -2 \end{pmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

The inverse matrix of the conic is:

$$\begin{pmatrix} 0 & 1/2 & -2 \\ 1/2 & 8 & -9/2 \\ -2 & -9/2 & -2 \end{pmatrix}^{-1} = \frac{1}{90} \begin{pmatrix} 145 & -40 & -55 \\ -40 & 16 & 4 \\ -55 & 4 & 1 \end{pmatrix}$$

So the equation of the tangential conic is:

$$\begin{bmatrix} 1-u-v & u & v \end{bmatrix} \begin{pmatrix} 145 & -40 & -55 \\ -40 & 16 & 4 \\ -55 & 4 & 1 \end{pmatrix} \begin{bmatrix} 1-u-v \\ u \\ v \end{bmatrix} = 0$$

$$241u^2 + 256v^2 + 488uv - 370u - 400v + 145 = 0$$

11.7 If we take A, B and C as the base of the points, the coordinates in this base of every point X will be expressed as:

$$X = (x_A, x_B, x_C) \qquad\qquad x_A + x_B + x_C = 1$$

If P is the intersection point of the lines AE and BF, then A, P and E are aligned and also B, P and F, conditions which we may express in coordinates:

$$\begin{vmatrix} 1 & 0 & 0 \\ p_A & p_B & p_C \\ e_A & e_B & e_C \end{vmatrix} = 0 \quad \begin{vmatrix} 0 & 1 & 0 \\ p_A & p_B & p_C \\ f_A & f_B & f_C \end{vmatrix} = 0 \;\Rightarrow\; (p_A, p_B, p_C) = \frac{(e_C f_A, e_B f_C, e_C f_C)}{e_C f_A + e_B f_C + e_C f_C}$$

If Q is the intersection point of the lines AD and CF, then A, Q and D are aligned and also C, Q and F, conditions which we may express in coordinates:

$$\begin{vmatrix} 1 & 0 & 0 \\ q_A & q_B & q_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & 1 \\ q_A & q_B & q_C \\ f_A & f_B & f_C \end{vmatrix} = 0 \quad \Rightarrow \quad (q_A, q_B, q_C) = \frac{(d_B\ f_A, d_B\ f_B, d_C\ f_B)}{d_B\ f_A + d_B\ f_B + d_C\ f_B}$$

If R is the intersection point of the lines BD and CE, then B, R and D are aligned and also C, R and E, conditions which we may express in coordinates:

$$\begin{vmatrix} 0 & 1 & 0 \\ r_A & r_B & r_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & 0 & 1 \\ r_A & r_B & r_C \\ e_A & e_B & e_C \end{vmatrix} = 0 \quad \Rightarrow \quad (r_A, r_B, r_C) = \frac{(d_A\ e_A, d_A\ e_B, d_C\ e_A)}{d_A\ e_A + d_A\ e_B + d_C\ e_A}$$

The points P, Q and R will be aligned if and only if the determinant of their coordinates vanishes. Setting aside the denominators, we have the first step:

$$\begin{vmatrix} p_A & p_B & p_C \\ q_A & q_B & q_C \\ r_A & r_B & r_C \end{vmatrix} \propto \begin{vmatrix} e_C f_A & e_B f_C & e_C f_C \\ d_B f_A & d_B f_B & d_C f_B \\ d_A e_A & d_A e_B & d_C e_A \end{vmatrix} \propto \begin{vmatrix} \frac{f_A}{f_C} & \frac{e_B}{e_C} & 1 \\ \frac{f_A}{f_B} & 1 & \frac{d_C}{d_B} \\ 1 & \frac{e_B}{e_A} & \frac{d_C}{d_A} \end{vmatrix} \propto \begin{vmatrix} \frac{1}{f_C} & \frac{1}{e_C} & \frac{1}{d_C} \\ \frac{1}{f_B} & \frac{1}{e_B} & \frac{1}{d_B} \\ \frac{1}{f_A} & \frac{1}{e_A} & \frac{1}{d_A} \end{vmatrix}$$

In the second step each row has been divided by a product of two coordinates. In the third step each column has been divided by a coordinate. And finally, after transposition and exchange of the first and third columns (turning the elements 90º in the matrix) and multiplying each column by a product of three coordinates we obtain:

$$\begin{vmatrix} p_A & p_B & p_C \\ q_A & q_B & q_C \\ r_A & r_B & r_C \end{vmatrix} \propto \begin{vmatrix} \frac{1}{f_A} & \frac{1}{f_B} & \frac{1}{f_C} \\ \frac{1}{e_A} & \frac{1}{e_B} & \frac{1}{e_C} \\ \frac{1}{d_A} & \frac{1}{d_B} & \frac{1}{d_C} \end{vmatrix} \propto \begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A f_A & d_B f_B & d_C f_C \\ e_A f_A & e_B f_B & e_C f_C \end{vmatrix} = 0$$

The last determinant is zero if and only if the point F lies on the conic passing through A, B, C, D and E because then it fulfils the conic equation:

$$\begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A x_A & d_B x_B & d_C x_C \\ e_A x_A & e_B x_B & e_C x_C \end{vmatrix} = 0$$

11.8 The Brianchon's theorem is the dual of the Pascal's theorem. In the dual plane, the point conic is represented by the tangential conic, the lines tangent to the point conic are represented by points lying on the tangential conic and the diagonals of the circumscribed hexagon are represented by the points of intersection P, Q and R of the former exercise. So the algebraic deduction is the same but in the dual plane.

12. Matrix representation and hyperbolic numbers

12.1 Let us calculate the modulus and argument:

$$|\, 5+4\,e_1| = \sqrt{5^2-4^2} = 3 \qquad \arg(5+4\,e_1) = \frac{1}{2}\arg\operatorname{tgh}\frac{4}{5} = \log 3$$

Then:

$$\sqrt{5+4\,e_1} = \sqrt{3}_{\frac{\log 3}{2}} = \sqrt{3}\left(\cosh\left(\log\sqrt{3}\right)+e_1\,\sinh\left(\log\sqrt{3}\right)\right) = 2+e_1$$

The four square roots are:

$$2+e_1 \qquad 1+2\,e_1 \qquad -2-e_1 \qquad -1-2e_1$$

12.2 $$2\,z^2+3\,z-17+3\,e_1=0 \qquad \Rightarrow \qquad z=\frac{-3\pm\sqrt{9-4\cdot 2\cdot(-17+3\,e_1)}}{2\cdot 2}$$

We must calculate the square root of the discriminant:

$$\sqrt{145-24\,e_1} = \sqrt{143}_{\frac{\log(11/13)}{2}} = 12-e_1$$

So the four solutions are:

$$z_1=\frac{9-e_1}{4} \qquad z_2=\frac{-15+e_1}{4} \qquad z_3=\frac{-1-6\,e_1}{2} \qquad z_4=-1+3\,e_1$$

12.3 Let us apply the new formula for the second degree equation:

$$z^2-6\,z+5=0 \qquad z=\frac{6\pm\sqrt{36-4\cdot1\cdot5}}{2\cdot1}=\frac{6\pm4}{2} \qquad \text{and}$$

$$z=\frac{6\pm e_1\sqrt{36-4\cdot1\cdot5}}{2\cdot1}=\frac{6\pm4\,e_1}{2}$$

12.4 $$\sin(x+y\,e_1)=\frac{1+e_1}{2}\sin(x+y)+\frac{1-e_1}{2}\sin(x-y)$$

$$=\sin x\cos y+e_1\,\cos x\sin y$$

12.5 From the analogous of Moivre's identity we have:

$$(\cosh\psi + e_1 \sinh\psi)^4 \equiv \cosh 4\psi + e_1 \sinh 4\psi$$

$$\cosh 4\psi \equiv \cosh^4\psi + 6\cosh^2\psi \sinh^2\psi + \sinh^4\psi$$

$$\sinh 4\psi \equiv 4\cosh^3\psi \sinh\psi + 4\cosh\psi \sinh^3\psi$$

12.6 The analytical continuation of the real logarithm is:

$$\log(x + y\, e_1) = \frac{1+e_1}{2}\log(x+y) + \frac{1-e_1}{2}\log(x-y)$$

$$= \frac{1}{2}\log(x^2 - y^2) + \frac{e_1}{2}\log\frac{x+y}{x-y}$$

It may be rewritten in the form:

$$= \log\sqrt{x^2 - y^2} + e_1 \operatorname{arg\,tgh}\frac{y}{x}$$

12.7 For the straight path $z = t\, e_1$ we have:

$$\int_{-e_1}^{e_1} z^2 dz = e_1 \int_{-1}^{1} t^2 dt = e_1 \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2\, e_1}{3}$$

For the circular path $z = \cos t + e_1 \sin t$ we have:

$$\int_{-e_1}^{e_1} z^2 dz = \int_{-\pi/2}^{\pi/2} (\cos t + e_1 \sin t)^2 (-\sin t + e_1 \cos t)\, dt$$

$$= \left[\cos t + \frac{2}{3}\cos^3 t + e_1\left(\sin t - \frac{2}{3}\sin^3 t\right)\right]_{-\pi/2}^{\pi/2} = \frac{2\, e_1}{3}$$

Using the indefinite integral, we find also the same result:

$$\int_{-e_1}^{e_1} z^2 dz = \left[\frac{z^3}{3}\right]_{-e_1}^{e_1} = \frac{2\, e_1}{3}$$

12.8 The proof is analogous to the exercise 3.12: turn the hyperbolic numbers *df*, *dz* into hyperbolic vectors by multiplying them at the left by e_2.

13. The hyperbolic or pseudo-Euclidean plane

13.1 If the vertices of the triangle are $A = (2, 2)$, $B = (1, 0)$ and $C = (5, 3)$ then:

$$AB = (-1, -2) \qquad BC = (4, 3) \qquad CA = (-3, -1)$$

$$|AB| = \sqrt{3}\, e_{12} \qquad |BC| = \sqrt{7} \qquad |CA| = \sqrt{8}$$

From the cosine theorem we have:

$$AB^2 = BC^2 + CA^2 - 2\,|BC|\,|CA| \cosh\gamma \quad \Rightarrow \quad \cosh\gamma = \frac{9}{2\sqrt{14}} \quad \Rightarrow \quad \gamma \cong 0.6264$$

taking into account that $\gamma = ACB$ is a positive angle. From the sine theorem we have:

$$\frac{|BC|}{\sinh\alpha} = \frac{|CA|}{\sinh\beta} = \frac{|AB|}{\sinh\gamma} \quad \Rightarrow \quad \frac{\sqrt{7}}{\sinh\alpha} = \frac{\sqrt{8}}{\sinh\beta} = \frac{\sqrt{3}\, e_{12}}{\frac{5}{2\sqrt{14}}}$$

whence the angles α and β follow:

$$\sinh\alpha = -\frac{5\, e_{12}}{2\sqrt{6}} \quad \Rightarrow \quad \alpha \cong -0.2027 - \frac{\pi}{2} e_{12}$$

$$\sinh\beta = -\frac{5\, e_{12}}{\sqrt{21}} \quad \Rightarrow \quad \beta \cong -0.4236 - \frac{\pi}{2} e_{12}$$

The plot helps to choose the right sign of the angles. Anyway a wrong choice would be revealed by the cosine theorem:

$$BC^2 = CA^2 + AB^2 - 2\,|CA|\,|AB| \cosh\alpha$$

$$CA^2 = AB^2 + BC^2 - 2\,|AB|\,|BC| \cosh\beta$$

13.2 The rotation of the vector is obtained with the multiplication by a hyperbolic number:

$$v' = v\, z_\alpha = (2\, e_2 + e_{21})(\cosh\log 2 + e_1 \sinh\log 2) = (2\, e_2 + e_{21})\left(\frac{5}{4} + \frac{3}{4} e_1\right)$$
$$= \frac{13}{4} e_2 + \frac{11}{4} e_{21}$$

The reflection is obtained by multiplying on the right and on the left by the direction vector of the reflection and its inverse respectively:

$$v' = d^{-1} v\, d = (3\, e_2 - e_{21})^{-1}\,(2\, e_2 + e_{21})(3\, e_2 - e_{21}) = \frac{1}{8}(3\, e_2 - e_{21})(7 - 5\, e_1) = \frac{13\, e_2 - 11\, e_{21}}{4}$$

The inversion with radius $r = 3$ is:

$$v' = r^2 v^{-1} = 9\,(2\,e_2 + e_{21})^{-1} = 3\,(2\,e_2 + e_{21}) = 6\,e_2 + 3\,e_{21}$$

13.3 The two-point equation of the line $y = 2\,x + 1$ is:

$$\frac{x+1}{1} = \frac{y}{2}$$

which gives us the direction vector:

$$v = e_1 + 2\,e_2$$

The direction vector is the normal vector of the perpendicular line. The general equation of a line in the hyperbolic plane is:

$$n \cdot PR = 0 \qquad \Rightarrow \qquad n_x(x - x_P) - n_y(y - y_P) = 0$$

Note the minus sign since this line lies on the hyperbolic plane. The substitution of the components of the direction vector of the first line and the coordinates of the point R = (3, 1) through which the line passes results in the equation:

$$1 \cdot (x-3) - 2 \cdot (y-1) = 0 \qquad \Rightarrow \qquad x - 2\,y - 1 = 0$$

The fact that both lines and the first quadrant bisector intersect in a unique point is circumstantial. Only it is required that the bisectors of both lines be parallel to the quadrant bisectors.

13.4 The line y =3 intersects the hyperbola $x^2 - y^2 = 16$ in the points R = (–5, 3) and R' = (5, 3). Then the power of P = (–7, 3) is:

$$PR\ PR' = 2\,e_2 \cdot 12\,e_2 = 24$$

The line $y = -\,3\,x - 18$ cuts the hyperbola in the points T = (–5, –3) and T' = (–17/2, 15/2). Then the power of P is:

$$PT\ PT' = (2\,e_2 - 6\,e_{21}) \cdot \left(-\frac{3\,e_2}{2} + \frac{9\,e_{21}}{2} \right) = -3 + 27 = 24$$

The line $y = -3\,x\,/\,7$ cuts the hyperbola in the points:

$$S = \left(-\frac{7\sqrt{10}}{5}, \frac{3\sqrt{10}}{5} \right) \qquad \text{and} \qquad S' = \left(\frac{7\sqrt{10}}{5}, -\frac{3\sqrt{10}}{5} \right)$$

The power is:

$$PS\ PS' = \left(\left(-\frac{7\sqrt{10}}{5} + 7 \right) e_2 + \left(\frac{3\sqrt{10}}{5} - 3 \right) e_{21} \right) \cdot \left(\left(\frac{7\sqrt{10}}{5} + 7 \right) e_2 + \left(-\frac{3\sqrt{10}}{5} - 3 \right) e_{21} \right)$$

$$= \frac{147}{5} - \frac{27}{5} = 24$$

Also the substitution of the coordinates gives 24:

$$x_P^2 - y_P^2 - 16 = 49 - 9 - 16 = 24$$

13.5 From the law of sines we find:

$$\frac{|a|}{|b|} = \frac{\sinh\alpha}{\sinh\beta} \qquad \Rightarrow \qquad \frac{|a| + |b|}{|a| - |b|} = \frac{\sinh\alpha + \sinh\beta}{\sinh\alpha - \sinh\beta}$$

Introducing the identity for the addition and subtraction of sines we arrive at the law of tangents:

$$\frac{|a| + |b|}{|a| - |b|} = \frac{\operatorname{tgh}\dfrac{\alpha + \beta}{2}}{\operatorname{tgh}\dfrac{\alpha - \beta}{2}}$$

13.6 The first triangle has the vertices $A = (0, 0)$, $B = (5, 0)$, $C = (5, 3)$ and the sides:

$$AB = 5\ e_2 \qquad BC = 3\ e_{21} \qquad CA = -5\ e_2 - 3\ e_{21}$$

The second triangle has the vertices $A' = (0, 0)$, $B' = (25, -15)$, $C' = (16, 0)$ and the sides:

$$A'B' = 25\ e_2 - 15\ e_{21} \qquad B'C' = -9\ e_2 + 15\ e_{21} \qquad C'A' = -16\ e_2$$

Both triangles are directly similar since:

$$AB^{-1}\ A'B' = BC^{-1}\ B'C' = CA^{-1}\ C'A' = 5 - 3\ e_1 = r$$

which is the similarity ratio r. The size ratio is the modulus of the similarity ratio:

$$|r| = \sqrt{5^2 - 3^2} = 4$$

that is, the second triangle is four times larger than the first. The angle of rotation between both figures is the argument of the similarity ratio:

$$\psi = \arg(5 - 3\ e_1) = \arg\operatorname{tgh}\left(-\frac{3}{5}\right) = -\log 2 = -0.6931...$$

Now plot the vertices of each triangle in the hyperbolic plane and break your Euclidean illusions about figures with the same shape.

14. Spherical geometry in the Euclidean space

14.1 The shadow of a gnomon follows a hyperbola on a plane (a quadrant) because the Sun describes approximately a parallel around the North pole during a day. The altitude of this parallel from the celestial equator is the declination, which changes slowly from one day to another. The March 21 and September 23 are the equinoxes when the Sun follows the equator. Since it is a great circle, its central projection is a straight line named the *equinoctial line*.

14.2 From the stereographic coordinates we find the Cartesian coordinates of the points:

$$A = -\frac{2}{3}e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 \qquad B = -\frac{12\,e_2}{13} + \frac{5\,e_3}{13} \qquad C = \frac{12}{13}e_1 + \frac{5}{13}e_3$$

Their outer products are equal to the cosines of the sides:

$$\cos a = B \cdot C = \frac{25}{169} \qquad a = 1.4223$$

$$\cos b = C \cdot A = -\frac{19}{39} \qquad b = 2.0797$$

$$\cos c = A \cdot B = -\frac{19}{39} \qquad c = 2.0797$$

Now we calculate the bivectors of the planes containing the sides:

$$A \wedge B = \frac{22\,e_{23} + 10\,e_{31} + 24\,e_{12}}{39} \qquad |\,A \wedge B\,| = \frac{\sqrt{1160}}{39}$$

$$B \wedge C = \frac{-60\,e_{23} + 60\,e_{31} + 144\,e_{12}}{169} \qquad |\,B \wedge C\,| = \frac{\sqrt{27936}}{169}$$

$$C \wedge A = \frac{-10\,e_{23} - 22\,e_{31} + 24\,e_{12}}{39} \qquad |\,C \wedge A\,| = \frac{\sqrt{1160}}{39}$$

The cosines of the angles between planes are obtained through the analogous of scalar product for bivectors:

$$\cos\alpha = -\frac{(C \wedge A)\cdot(A \wedge B)}{|\,C \wedge A\,||\,A \wedge B\,|} = -\frac{136}{1160} \qquad \alpha = 1.6882$$

$$\cos\beta = -\frac{(A \wedge B)\cdot(B \wedge C)}{|\,A \wedge B\,||\,B \wedge C\,|} = -\frac{2736}{\sqrt{1160 \cdot 27936}} \qquad \beta = 2.0721$$

$$\cos\gamma = -\frac{(B\wedge C)\cdot(C\wedge A)}{|\,B\wedge C\,||\,C\wedge A\,|} = -\frac{2736}{\sqrt{27936\cdot 1160}} \qquad \gamma = 2.0721$$

The area of the triangle is the spherical excess:

$$area = \alpha + \beta + \gamma - \pi = 2.6908$$

14.3 Let us calculate the Cartesian coordinates of Fastnet (point *F*), the point in the middle Atlantic (point *A*) and Sandy Hook (point *S*). Identifying the geographical longitude with the angle φ and the colatitude with the angle θ we have:

$$x = \sin\theta\cos\varphi \quad y = \sin\theta\sin\varphi \qquad z = \cos\theta$$

$$F = (\theta = 38^\circ 30', \varphi = -9^\circ 35') = 0.6138\, e_1 - 0.1036\, e_2 + 0.7826\, e_3$$

$$A = (\theta = 47^\circ 30', \varphi = -47^\circ) = 0.5028\, e_1 - 0.5392\, e_2 + 0.6756\, e_3$$

$$S = (\theta = 49^\circ 30', \varphi = -74^\circ) = 0.2096\, e_1 - 0.7309\, e_2 + 0.6494\, e_3$$

The length of the arcs is obtained from the inner products of the position vectors:

$$\cos FS = F\cdot S = 0.7126 \qquad FS = 44^\circ 33' = 4950 \text{ km}$$

$$\cos FA = F\cdot A = 0.8932 \qquad FA = 26^\circ 43' = 2969 \text{ km}$$

$$\cos AS = A\cdot S = 0.9382 \qquad AS = 20^\circ 14' = 2250 \text{ km}$$

Now we see that the track is 269 km longer than the shortest path between Fastnet and Sandy Hook.

Let us investigate whether the Titanic followed the obliged track, that is whether the point *T* where the tragedy happened lies on the line *AS*, by calculating the determinant of the three points:

$$T = (\theta = 48^\circ 14', \varphi = -50^\circ 14') = 0.4771\, e_1 - 0.5733\, e_2 + 0.6661\, e_3$$

$$A\wedge T\wedge S = \begin{vmatrix} 0.5028 & 0.4771 & 0.2096 \\ -0.5392 & -0.5733 & -0.7309 \\ 0.6756 & 0.6661 & 0.6494 \end{vmatrix} e_{123} = -0.0050\, e_{123}$$

The negative sign indicates that the orientation of the three points is clockwise (figure 16.6). In other words, the Titanic was shipping at the South of the obliged track. We can calculate the distance from *T* to the track *AS* taking the right angle triangle. By the Napier's rule we have:

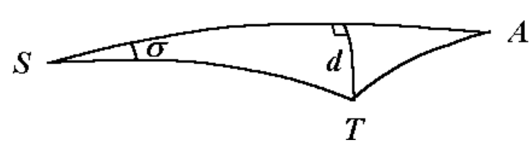


Figure 16.6

$$\sin d = \sin\sigma \sin ST = \frac{|A \wedge T \wedge S|}{|A \wedge S|} = \frac{0.0050}{0.3461} = 0.0144 \quad d = 49' = 92 \text{ km}$$

14.4 After removing the margins and subtracting the coordinates of the centre of the photograph in the bitmap file, I have obtained a pair of coordinates u' and v' in pixels, which are proportional to u and v:

star	u'	v'
α Cassiopeia	173	–139
β Cassiopeia	349	–135
γ Cassiopeia	199	24
δ Cassiopeia	89	93
Hale-Bopp comet	–240	–320

On the other hand, the right ascension A and declination D of these stars are known data. From the spherical coordinates and taking θ = 90º – D and φ = A one obtains their Cartesian equatorial coordinates in the following way:

$$x = \cos D \cos A \qquad y = \cos D \sin A \qquad z = \sin D$$

The Aries point has the equatorial coordinates (1,0,0). Using these formulas I have found the coordinates:

star	x	y	z
α Cassiopeia	0.543519	0.096098	0.833878
β Cassiopeia	0.512996	0.019711	0.858216
γ Cassiopeia	0.475011	0.119054	0.871889
δ Cassiopeia	0.462917	0.180840	0.867759

A photograph is a central projection. So the arch s between two stars A and B is related with their coordinates (u, v) on the projection plane by:

$$\cos s_{AB} = \frac{1 + u_A u_B + v_A v_B}{\sqrt{1 + u_A^2 + v_A^2}\sqrt{1 + u_B^2 + v_B^2}}$$

The focal distance f is the distance between the projection plane (the photograph) and the centre of projection. When the focal distance is unknown we only can measure proportional coordinates u' and v' instead of u and v, and the foregoing formula becomes:

$$\cos s_{AB} = \frac{f^2 + u'_A u'_B + v'_A v'_B}{\sqrt{f^2 + u'^2_A + v'^2_A}\sqrt{f^2 + u'^2_B + v'^2_B}}$$

which is a second degree equation on f^2:

$$\left(f^2 + u'^2_A + v'^2_A\right)\left(f^2 + u'^2_B + v'^2_B\right)\cos^2 s_{AB} = \left(f^2 + u'_A u'_B + v'_A v'_B\right)^2$$

$$0 = f^4\left(1 - \cos^2 s_{AB}\right) + f^2\left[2\left(u'_A u'_B + v'_A v'_B\right) - \left(u'^2_A + v'^2_A + u'^2_B + v'^2_B\right)\cos^2 s_{AB}\right]$$

$$+\left(u'_A u'_B + v'_A v'_B\right)^2 - \left(u'^2_A + v'^2_A\right)\left(u'^2_B + v'^2_B\right)\cos^2 s_{AB}$$

All the coefficients of the equation are known, because cos $s_{AB} = A \cdot B$ is calculated from the Cartesian equatorial coordinates obtained from the right ascension and declination of both stars. The focal distances so obtained are:

star	f
α-β Cassiopeia	2012.7
α-γ Cassiopeia	2010.6
α-δ Cassiopeia	2017.2
β-γ Cassiopeia	2003.0
β-δ Cassiopeia	2019.1
γ-δ Cassiopeia	2049.1
mean value	2018.6

The width of the photograph is 750 pixels and the height 1166. If 750 pixels corresponds to 24 mm and 1166 to 36 mm, we find a focal distance of the camera equal to 64.6 mm and 62.3 mm respectively. However the author of the photograph indicated me that his camera has a focal distance of 58 mm, what implies that the original image was cut a 7 % in the photographic laboratory. This is a customary usage in photography, so we do not know the enlargement proportion of a paper copy and we must calculate the focal distance directly from the photograph, which is always somewhat higher than that calculated from the focal distance of the camera. On the other hand, the calculus of the focal distance is very sensitive to the errors and truncation of decimals. The best performance is to write a program to calculate the mean focal distance from the original data or to make us the paper copies without cutting the image. Known the focal distance, we are able to calculate the normalised coordinates u and v:

star	u	v
α Cassiopeia	0.085703	–0.068860

β Cassiopeia	0.172892	–0.066878
γ Cassiopeia	0.098583	0.011889
δ Cassiopeia	0.044090	0.046072
Hale-Bopp comet	–0.118894	–0.158526

and from here the inner product between stars and the comet:

$$\cos(A,H)=\frac{1+u_A\,u_H+v_A\,v_H}{\sqrt{1+u_A^2+v_A^2}\sqrt{1+u_H^2+v_H^2}}$$

$$\cos(B,H)=\frac{1+u_B\,u_H+v_B\,v_H}{\sqrt{1+u_B^2+v_B^2}\sqrt{1+u_H^2+v_H^2}}$$

The unknown equatorial coordinates of the comet H are calculated from a system of three equations:

$$\begin{cases} A\cdot H=\cos(A,H) \\ B\cdot H=\cos(B,H) \\ \quad H^2=1 \end{cases}$$

Let us introduce the Clifford product instead of the inner product and multiply the first equation by B and the second one by A on the right:

$$\begin{cases} \dfrac{A\,H+H\,A}{2}=\cos(A,H) \;\Rightarrow\; \dfrac{A\,H\,B+H\,A\,B}{2}=\cos(A,H)\,B \\ \dfrac{B\,H+H\,B}{2}=\cos(B,H) \;\Rightarrow\; \dfrac{B\,H\,A+H\,B\,A}{2}=\cos(B,H)\,A \\ \qquad H^2=1 \end{cases}$$

Using the permutative property (in the space $u\ v\ w-w\ v\ u=2\ u\wedge v\wedge w$), the difference of both equations is equal to:

$$A\wedge H\wedge B+H\ A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$$

or equivalently: $\quad -H\wedge A\wedge B+H\ A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$

The outer product by H is the geometric product by its perpendicular component $H_\perp$, so we have:

$$H_\parallel\ A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$$

where we can isolate the coplanar component:

$$H_{\parallel} = \left(\cos(A,H)\,B - \cos(B,H)\,A\right)\left(A\wedge B\right)^{-1}$$

Observe that this result can be only obtained and expressed through the geometric product, and not through the more usual inner and outer products. The component perpendicular to the plane *AB* is proportional to the dual vector of the outer product of this vectors. Known the coplanar component we may now fix the modulus of the perpendicular component because $H^2=1$:

$$H_{\perp} = \pm\frac{e_{123}\;A\wedge B}{|\,A\wedge B\,|}\sqrt{1-H_{\parallel}^2}$$

There are two solutions, but according to the statement of the problem only one of them is valid. Since each pair of stars gives two values of *H* (the position vector of the comet), we may distinguish the true solution because it has proper values for different pairs of stars. The mean value with the standard deviation so obtained is:

$$H = \left(0.661606\pm 0.000741\right)e_1 + \left(0.236258\pm 0.001298\right)e_2 + \left(0.711659\pm 0.000302\right)e_3$$

from where the following equatorial coordinates of the Hale-Bopp comet are obtained:

$$A_H = 1\text{h}\;18\text{min}\;36\text{s} \qquad\qquad D_H = 45^{\text{o}}22'12''$$

In the ephemeris (http://www.xtec.es/recursos/astronom/hb/ephjan97.txt) for the comet we find:

Date	Right ascension (*A*)	Declination (*D*)
1997 March 28	01h 09min 18.03s	+45º 36’ 30.6”
1997 March 29	01h 19min 06.07s	+45º 25’ 15.6”
1997 March 30	01h 28min 44.92s	+45º 10’ 51.9”

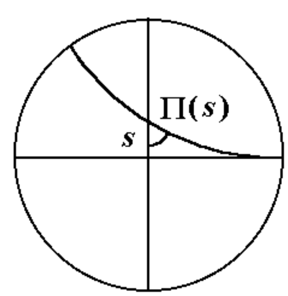
So I conclude that this photograph was taken the March 29 of 1997.

On the other hand, I have also calculated the orientation *O* of the camera (of the line passing through the centre of the photograph):

$$O = \left(0.513505\pm 0.000673\right)e_1 + \left(0.201570\pm 0.000471\right)e_2 \pm \left(0.834075\pm 0.000360\right)e_3$$

$$A_O = 1\text{h}\;25\text{min}\;44\text{s} \qquad\qquad D_O = 56^{0}31'11''$$

15. Hyperboloidal geometry in the pseudo-Euclidean space (Lobachevsky’s geometry)



15.1 The circle being projection of a parallel line (geodesic hyperbola) shown in the figure 15.9 passes through the point (1, 0) which implies the equation:

$$(u-1)^2 + (v-b)^2 = b^2$$

According to this equation, the intersection of this line with the v-axis has the coordinates $\left(0, b-\sqrt{b^2-1}\right)$ (the other intersection point falls outside the Poincaré disk). The inverse of the slope at this point with opposite sign is the tangent of the angle of parallelism:

Figure 15.9

$$\operatorname{tg}\Pi(s) = -\left(\frac{du}{dv}\right)_{u=0} = \sqrt{b^2-1} \qquad \Rightarrow \qquad \cos\Pi(s) = \frac{1}{b}$$

Now let us calculate the distance s from the origin to this point. Its hyperbolic cosine is the inner product of the unitary position vectors of the origin and the point of intersection:

$$\cosh s = e_3 \cdot \left(\left(\frac{b-\sqrt{b^2-1}}{-b^2+1+b\sqrt{b^2-1}} \right) e_2 + \frac{b}{\sqrt{b^2-1}}\, e_3 \right) = \frac{b}{\sqrt{b^2-1}}$$

$$s = \log\sqrt{\frac{b+1}{b-1}} \quad \exp(-s) = \sqrt{\frac{b-1}{b+1}} = \sqrt{\frac{1-\cos\Pi(s)}{1+\cos\Pi(s)}} = \operatorname{tg}\frac{\Pi(s)}{2}$$

so we find the Lobachevsky's formula for the angle of parallelism:

$$\Pi(s) = 2\operatorname{arctg}(\exp(-s))$$

15.2 The law of cosines in hyperboloidal trigonometry is:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos a$$

For a right angle triangle $\alpha = \pi/2$:

$$\cosh a = \cosh b \cosh c$$

In the limit of small arcs it becomes the Pythagorean theorem:

$$1 + \frac{a^2}{2} + O(a^4) = \left(1 + \frac{b^2}{2} + O(b^4)\right)\left(1 + \frac{c^2}{2} + O(c^4)\right)$$

$$a^2 + O(a^4) = b^2 + c^2 + O(b^2c^2, b^4, c^4)$$

15.3 A «circle» is given by the equation system:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ a\,x + b\,y + c = z \end{cases}$$

a) The intersection of a quadric (hyperboloid) with a plane is always a conic. Since the curve is closed, it must be an ellipse (for horizontal planes it is a circle).
b) We hope that the centre of the ellipse be the intersection of the plane with its perpendicular line passing through the origin (this occurs also in spherical geometry). The plane has a bivector:

$$a\,e_{23} + b\,e_{31} - e_{12}$$

The vector perpendicular, the dual vector is:

$$a\,e_1 + b\,e_2 + e_3$$

because the product of the vector and bivector is equal to a pure volume element:

$$(a\,e_{23} + b\,e_{31} - e_{12})(a\,e_1 + b\,e_2 + e_3) = \left(a^2 + b^2 - 1\right)e_{123}$$

The *axis* of a circle is the line passing through the origin and perpendicular to the plane containing this circle. The centre of the circle is the intersection point of the plane and axis of this circle, given by the equation system:

$$\begin{cases} a\,x + b\,y + c = z \\ \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{1} \end{cases} \Rightarrow \quad x = \frac{a\,c}{1 - a^2 - b^2} \qquad y = \frac{b\,c}{1 - a^2 - b^2} \qquad z = \frac{c}{1 - a^2 - b^2}$$

The distance from any point of the circle to its centre is constant:

$$\left(z - \frac{c}{1 - a^2 - b^2}\right)^2 - \left(x - \frac{a\,c}{1 - a^2 - b^2}\right)^2 - \left(y - \frac{b\,c}{1 - a^2 - b^2}\right) =$$

$$1 - \frac{2\,c^2}{1 - a^2 - b^2} + \frac{c^2\left(1 - a^2 - b^2\right)}{\left(1 - a^2 - b^2\right)^2} = 1 - \frac{c^2}{1 - a^2 - b^2}$$

Since $c > 1$ and $a^2 + b^2 < 1$ this value is always negative, what means that the radius is a distance comparable with distances on the Euclidean plane *x*-*y*. Taking the real value, the radius of a circle is:

$$r = \sqrt{\frac{c^2}{1 - a^2 - b^2} - 1}$$

However this is not the radius measured on the hyperboloid and obtained from its projections. The intersection of the axis with the hyperboloid gives its hyperboloidal centre:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{1} \end{cases} \Rightarrow \quad x = \frac{a}{1-a^2-b^2} \qquad y = \frac{b}{1-a^2-b^2} \qquad z = \frac{1}{1-a^2-b^2}$$

The hyperboloidal radius is the arc length on the hyperboloid from any point of the circle to its hyperboloidal centre, which should be constant. Making the inner product of the position vectors of a point on the circle and its centre, we find:

$$\cosh\psi = (x\, e_1 + y\, e_2 + z\, e_3)\cdot\left(\frac{a}{1-a^2-b^2}e_1 + \frac{b}{1-a^2-b^2}e_2 + \frac{z}{1-a^2-b^2}e_3\right) =$$
$$\frac{-a\,x - b\,y + z}{1-a^2-b^2} = \frac{c}{1-a^2-b^2}$$

So the hyperboloidal radius is:

$$\psi = \arg\cosh\frac{c}{1-a^2-b^2}$$

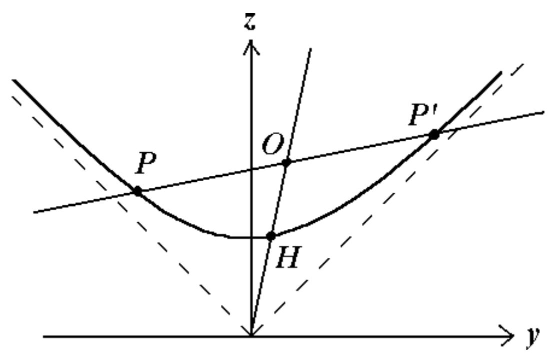


Figure 16.7

The figure 16.7 shows a lateral view of a plane with $a = 0$ and equation $b\,x + c = z$. The points P and P' lie on the circle. O is the centre of the circle on its plane and H is the centre on the hyperboloid. Then OH is the axis of the circle, which is perpendicular (in a pseudo-Euclidean way) to the circle plane, that is, their bisector has a unity slope. Note that according to a known property of the hyperbola, O must be the midpoint of the chord PP'. The plane radius is the distance PO or $P'O$, while the hyperboloidal radius is the arc length PH or $P'H$.

c) The coordinates of the stereographic projection are:

$$x = \frac{2\,u}{1-u^2-v^2} \qquad y = \frac{2\,v}{1-u^2-v^2} \qquad z = \frac{1+u^2+v^2}{1-u^2-v^2}$$

Let us make the substitution of these coordinates in the equation of the circle plane:

$$\frac{2\,a\,u + 2\,b\,v}{1-u^2-v^2} + c = \frac{1+u^2+v^2}{1-u^2-v^2}$$

After simplification we arrive to an equation of a circle:

$$\left(u - \frac{a}{c+1}\right)^2 + \left(v - \frac{b}{c+1}\right)^2 = \frac{a^2+b^2+c^2-1}{(c+1)^2}$$

15.4 A horocycle is a circle having $a^2 + b^2 = 1$. Therefore its equation in the stereographic projection is:

$$\frac{c^2}{(c+1)^2} = \left(u - \frac{a}{c+1}\right)^2 + \left(v - \frac{b}{c+1}\right)^2$$

Let us search the intersection points (if they exist) with the limit circle $u^2 + v^2 = 1$. Then we solve the system of both equations and find that it has a unique solution:

$$u = a \qquad\qquad v = b$$

Since they meet in a unique point and $a^2 + b^2 = 1$, the horocycle projection is tangent to the limit circle. The centre of the horocycle is the intersection of the hyperboloid with its axis. However this axis has unity slope, so the intersection lies at the infinity.

15.5 By differentiation of the coordinates in the Beltrami projection we find:

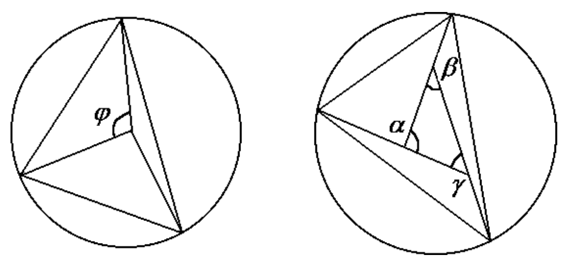
$$dx = \frac{\left(1 - v^2\right)du + u\,v\,dv}{\left(1 - u^2 - v^2\right)^{3/2}} \qquad dy = \frac{\left(1 - u^2\right)dv + u\,v\,du}{\left(1 - u^2 - v^2\right)^{3/2}}$$

$$dz = \frac{u\,du + v\,dv}{\left(1 - u^2 - v^2\right)^{3/2}}$$

The differential of area is a bivector and we search its modulus:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{du \wedge dv}{\left(1 - u^2 - v^2\right)^{3/2}}$$

Note the outer products of the differentials of coordinates so $dx \wedge dy = -dy \wedge dx$. Now we must integrate the differential of area of a doubly asymptotic triangle with angle φ (figure 15.10a). We change to polar coordinates r and θ in the Beltrami disk:



$$r = \sqrt{u^2 + v^2} \qquad\qquad \operatorname{tg}\theta = \frac{v}{u}$$

Figure 15.10

Since the equation of the line is $r = \dfrac{\cos(\varphi/2)}{\cos\theta}$ the integral becomes:

$$A = \int_0^{\frac{\cos(\varphi/2)}{\cos\theta}} \int_{-\varphi/2}^{\varphi/2} \frac{r\,dr \wedge d\theta}{\left(1 - r^2\right)^{3/2}} = \int_{-\varphi/2}^{\varphi/2} \left[\frac{1}{\sqrt{1 - r^2}}\right]_0^{\frac{\cos(\varphi/2)}{\cos\theta}} d\theta = \int_{-\varphi/2}^{\varphi/2} \frac{\cos\theta}{\sqrt{1 - \cos^2\frac{\varphi}{2} - \sin^2\theta}}\,d\theta - \int_{-\varphi/2}^{\varphi/2} d\theta =$$

$$= \left[\arcsin \frac{\sin\theta}{\sin\frac{\varphi}{2}} - \theta \right]_{-\varphi/2}^{\varphi/2} = \pi - \varphi$$

This expression is applied to each asymptotic triangle surrounding the central triangle (figure 15.10b). Since the area of a triangle having the three vertices in the limit circle is $3\pi - 2\pi = \pi$, the area of the central triangle follows:

$$A = \pi - \alpha - \beta - \gamma$$

15.6 In the azimuthal projection, the radius r of a circle centred at the origin is:

$$u = x\sqrt{\frac{2}{z+1}} \qquad v = y\sqrt{\frac{2}{z+1}} \qquad r^2 = u^2 + v^2 = 2\left(z-1\right)$$

Since the projection is equivalent, we can evaluate the area directly in the plane:

$$A = \pi\, r^2 = 2\,\pi\left(z-1\right) = 2\,\pi\left(\cosh\psi - 1\right)$$

where ψ is the Weierstrass coordinate, that is, the radius of the circle on the hyperboloid.

BIBLIOGRAPHY

R. ABLAMOWICZ, P. LOUNESTO, J. M. PARRA eds. *Clifford Algebras with numeric and symbolic computation applications.* Birkhäuser (Boston, 1996).

D. V. ALEKSEEVSKIJ, E. B. VINBERG, A. S. SOLODOVNIKOV. I.*Geometry of Spaces of Constant Curvature* in *Geometry* vol. II, *Spaces of Constant Curvature*, ed. by E. B. Vinberg. *Encyclopaedia of Mathematical Sciencies* **29**, Springer Verlag (New York, 1993).

William E. BAYLIS (editor). *Clifford (Geometric) Algebras*. Birkhäuser (1996).

Marcel BERGER.. *Geometry*, vol. I-II. Springer-Verlag (Berlin, Heidelberg,1987).

Roberto BONOLA. *Non-Euclidean Geometry*. Dover Publications Inc. (New York, 1955).

Alfred M. BORK. « "Vectors Versus Quaternions"—The letters in *Nature*», *American Journal of Physics*, **34** (1966) 202-211.

F. BRACKX, R. DELANGHE, H. SERRAS eds. *Clifford Algebras and Their Applications in Mathematical Physics*. Kluwer (Dordrecht, 1993).

Lev M. BUGAYEVSKIY, John P. SNYDER. *Map projections. A reference manual*. Taylor & Francis (London, 1995).

David A. BRANNAN, Matthew F. ESPLEN, Jeremy J. GRAY. *Geometry*. Cambridge University Press (Cambridge, 1999).

Gaston CASANOVA. *L'algèbre vectorielle*. Presses Universitaires de France (París, 1976).

G. CASANOVA. «Fonctions analytiques hyperboliques», *Advances in Applied Clifford Algebras*, **9** (1999) 93.

G. CASANOVA. «Roots of algebraic equations and Clifford algebra», *Advances in Applied Clifford Algebras*, **9** (1999) 215.

William Kingdon CLIFFORD. «Applications of Grassmann's Extensive Algebra» *American Journal of Mathematics Pure and Applied*, **1** (1878) 350-358, reproduced in *Mathematical papers by William Kingdon Clifford,* 266-276. Chelsea Publishing Company (New York, 1968).

W. K. CLIFFORD. «On the classification of geometric algebras», *Mathematical papers by William Kingdon Clifford*, 397-401. Chelsea Publishing Company (New York, 1968).

Julian Lowell COOLIDGE. *A History of the Conic Sections and Quadric Surfaces*. Dover Pub. Inc. (New York, 1968).

Harold S. M. COXETER. *Introduction to Geometry*. John Wiley & Sons, Inc. (NewYork, 1961).

H. S. M. COXETER. *The Real Projective Plane*, 2nd ed. Cambridge Univ. Press (Cambridge, 1955).

Michael J. CROWE. *A History of Vector Analysis. The Evolution of the Idea of a Vectorial System*. Dover Publications, Inc. (New York, 1994).

N. V. EFIMOV. *Higher geometry.* Mir Publishers (Moscow, 1980).

Howard EVES, *A Survey of Geometry*. Allyn and Bacon, Inc. (London, 1965).

Richard L. FABER. *Foundations of Euclidean and Non-Euclidean Geometry*. Marcel Dekker, Inc. (New York, 1983).

Werner FENCHEL. *Elementary Geometry in Hyperbolic Space*. De Gruyter Studies in Mathematics **11**, de Gruyter (New York, 1989).

Henry G. FORDER. *The Calculus of Extension*. Cambridge University Press (London, 1941).

Rudolf FUCKE, Konrad KIRCH, Heinz NICKEL. *Darstellende Geometrie*. VEB Fachbuchverlag (Leipzig, 1987).

C. I. GERHARDT.«Leibniz an Huygens» in *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematiker*, 570.Georg Olms Verlag (Hildesheim, 1987).

C. I. GERHARDT. «Characteristica geometrica. Analysis geometrica propria. Calculus situs» in *G. W. Leibniz. Mathematische Schriften* **V**, 141-183. Georg Olms Verlag (Darmstadt, 1971).

Josiah Willard GIBBS. «On multiple algebra». *Proceedings of the American Association for the Advancement of Science* **XXXV** (1886) 37-66. Reproduced together with the following letters to *Nature* in *Scientific works of J. W. Gibbs*, **II**, Longman, Green (London, 1906).

J. W. GIBBS. «On the role of quaternions in the algebra of vectors». *Nature*, **XLIII** (1891)

511-513.

J. W. GIBBS. «Quaternions and the *Ausdehnungslehre*». *Nature*, **XLIV** (1891) 79-82.

J. W. GIBBS. «Quaternions and the algebra of vectors». *Nature*, **XLVII** (1893) 463-464.

J. W. GIBBS. «Quaternions and vector analysis». *Nature*, **XLVIII** (1893) 364-367.

J. W. GIBBS, Edwin Bidwell WILSON (author). *Vector analysis, a text-book for the use of students of mathematics and physics founded upon the lectures of J. Willard Gibbs*. Yale bicentennial publications, C. Scribner's Sons (New York, 1901).

Ramon GONZALEZ, Josep HOMS, Jordi SOLSONA. *Butlletí de la Societat Catalana de Matemàtiques* **12, 1** (1997) 51-71 (English translation «Estimate of the error committed in determining the place from where a photograph has been taken» at http://www.terra.es/personal/rgonzal1/fotoart2.htm).

Hermann GRASSMANN. *Die Ausdehnungslehre*, first edition (Szczecin, 1844). Spanish translation *Teoría de la extensión*[1] by Emilio Óscar Roxín in the collection *Historia y filosofía de la ciencia*, Espasa-Calpe (Buenos Aires, 1947).

H. GRASSMANN. *A new branch of mathematics: The "Ausdehnungslehre" of 1844 and other works*. Translated by Lloyd C. Kannenberg. Open Court (Chicago, 1995).

H. GRASSMANN. *Extension Theory*. Translation of *Die Ausdehnungslehre* of 1862 by Lloyd C. Kannenberg. American Mathematical Society (Providence, Rhode Island, 2000).

H. GRASSMANN. «The position of the Hamiltonian quaternions in the extension theory». Translation of «Der Ort der Hamilton'schen Quaternionen in der Ausdenungslehre», *Mathematik Annalen* **12** (1877) 375-386, by Lloyd C. Kannenberg in *A new branch of mathematics: The "Ausdehnungslehre of 1844 and other works*, 525-538 and editorial notes 539-542.

V. GUSEV, V. LITVINENKO, A. MORDKOVICH. *Solving Problems in Geometry*. Mir Publishers (Moscow, 1988).

William Rowan HAMILTON. *Elements of Quaternions* (1869). Ed. by Charles Jasper Joly, 3rd edition, Chelsea Publishing Company (New York, 1969).

Oliver HEAVISIDE. «The elements of vectorial algebra and analysis» (13th Nov 1891), *Electromagnetic Theory, The Electrician Series* (1891-1898), The Electrician printing and publishing Company Ltd. (London).

David HESTENES. *Space-Time Algebra*. Gordon and Breach (New York, 1966).

D. HESTENES. *New Foundations for Classical Mechanics*. Ed. by Alwyn Van der Merwe. D. Reidel Publishing Company (Dordrecht, 1986).

D. HESTENES, Garret SOBCZYK. *Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics*, Reidel Publishing Company (Dordrecht, 1984).

D. HESTENES. «Mathematical viruses» in *Clifford Algebras and their Applications in Mathematical Physics*, 3-16, edited by A. Micali, R. Boudet and J. Helmstetter. Kluwer Academic Publishers (Dordrecht, 1992).

David HILBERT. *Grundlagen der Geometrie. Mit Supplementen von Paul Bernays*. Teubner (Stuttgart, 1977).

David HILBERT, S. COHN-VOSSEN. *Geometry and the Imagination*. Chelsea Publishing Company (New York, 1952).

International Cartographic Association. *Basic Cartography for students and technician.* (Over Wallop, 1984).

Birger IVERSEN. *Hyperbolic Geometry*. London Mathematical Society Student Texts, **25**. Cambridge University Press (Cambridge, 1992).

George A. JENNINGS. *Modern Geometry with Applications*. Springer-Verlag (New York, 1994).

Felix KLEIN. *Vorlesungen über nicht-euklidische geometrie*. Verlag von Julius Springer

[1] In this edition, the date in the preface by H. Grassmann was misprinted as 1884 instead of 1844.

(Berlin, 1928).
Daniel LEHMANN, Rudolph BKOUCHE. *Initiation à la géométrie*. Presses universitaires de France (Paris, 1988).
Pertii LOUNESTO. *Clifford Algebras and Spinors*. Cambridge University Press (Cambridge, 1997).
George E. MARTIN. *The Foundations of Geometry and the Non-Euclidean Plane*. Springer-Verlag (New York, 1975).
Edwin E. MOISE. *Elementary Geometry from an Advanced Standpoint*, Addison-Wesley Publishing Company, 2nd ed. (Reading, Massachusetts, 1974)
Junjiro NOGUCHI. *Introduction to Complex Analysis*. Translations of Mathematical Monographs, **168**. American Mathematical Society (Providence, Rhode Island, 1998).
Josep Manel PARRA i SERRA. «Clifford algebras. Towards a common language for mathematicians and physicists. Some suggested notations», *Advances in Applied Clifford Algebras* **2** (1992) 145.
J. M. PARRA I SERRA. «Geometrical algebra versus numerical Cartesianism. The historical trend behind Clifford's algebra», in *Clifford Algebras and their Applications in Mathematical Physics*, 307-316. F. Brackx, R. Delanghe, H. Serras eds., Kluwer, Dordrecht (Deinze, 1993).
J. M. PARRA I SERRA. «Teaching Clifford Algebras as Physiscal Mathematics», in *Clifford Algebras and their Applications in Mathematical Physics*, 291-304. V. Dietrich, K. Habetha, G. Janks eds. Kluwer, Dordrecht (Aachen, 1996).
Giuseppe PEANO. *Calcolo geometrico secondo l'Ausdehnungslehre di Hermann Grassmann*. Fratelli Boca Editori (Torino, 1888).
G. PEANO. «Saggio di Calcolo geometrico», *Atti della Reale Accad. Delle Scienze di Torino* **XXXI** (1895-96) 952-975. Collected in *Opere Scelte*, **III**, 167, ed. Cremone, (Roma, 1959). Also collected and translated by Hubert C. Kennedy in *Selected Works of Giuseppe Peano*, 169-188. George Allen & Unwin Ltd. (London, 1973).
Dan PEDOE. *Geometry, A Comprehensive Course*. Dover Publications, Inc., (New York, 1988).
George POLYA, Gordon LATTA. *Complex variables*. John Wiley & Sons, Inc., (New York, 1974).
Murali RAO, Henrik STETKÆR. *Complex Analysis, an invitation*. World Scientific (Singapore, 1991).
Hans REICHARDT. *Gauss und die Anfänge der nicht-euklidischen Geometrie. Mit Originalarbeiten von J. Bolyai, N. I. Lobatschewski und F. Klein.* Teubner-Archiv zur Mathematik, **4**. BSB B. G. Teubner Verlaggesellschaft (Leipzig, 1985).
Marcel RIESZ. *Clifford Numbers and Spinors.* The Institute of Fluid Mechanics and Applied Mathematics, Univ. Maryland (1958). Reprinting E. Folke Bolinder, P. Lounesto eds. Kluwer (Dordrecht, 1993).
Boris A. ROSENFELD. *A History of Non-Euclidean Geometry. Evolution of the Concept of a Geometric Space*. Studies in the History of Mathematics and Physical Sciences, **12**. Springer-Verlag (New York, 1988).
Patrick J. RYAN. *Euclidean and non-Euclidean geometry. An analytic approach.* Cambridge University Press (Cambridge, 1986).
A. A. SAZANOV. *El universo tetradimensional de Minkowski*. Editorial Mir (Moscú, 1990).
I. SHARIGUIN. *Problems in plane geometry*. Mir Publishers (Moscow, 1988).
Hans SCHWERDTFEGER. *Geometry of Complex Numbers*. Dover Publications, Inc. (New York, 1979).
John STILLWELL. *Geometry of Surfaces*. Universitext, Springer-Verlag (New York, 1992).
The teaching of trigonometry in schools. A report prepared for the Mathematical Association. G. Bell & Sons, Ltd. (London, 1962).
C. ZWIKKER. *The advanced geometry of plane curves and their applications*. Dover Pub. Inc. (New York, 1963).

INTERNET BIBLIOGRAPHY

Personal pages

Rafal Ablamowicz http://math.tntech.edu/rafal/
William E. Baylis http://www.cs.uwindsor.ca/users/b/baylis/
Ramon González Calvet. http://www.terra.es/personal/rgonzal1
David Hestenes. http://modelingnts.la.asu.edu/
Jaime Keller http://www.tuwien.ac.at/cms/jkeller/
Pertii Lounesto. http://www.helsinki.fi/~lounesto/
Josep Manel Parra i Serra. http://hermes.ffn.ub.es/~jmparra

Software pages

CABRI http://www.cabri.com
CINDERELLA homepage http://www.cinderella.de/en/index.html
application to geometric algebra http://carol.wins.uva.nl/~leo/cinderella/
CLICAL (Clifford Calculator) http://www.helsinki.fi/~lounesto/CLICAL.htm
CLU http://www.perwass.de/cbup/start.html
GAIGEN http://carol.wins.uva.nl/~fontijne/gaigen/about.html
GEOMA http://nklein.com/products/geoma/
GLUCAT http://glucat.sourceforge.net/
MAPLE V package http://math.tntech.edu/rafal/cliff5/
NONEUCLID http://math.rice.edu/~joel/NonEuclid/
REDUCE packages http://hermes.ffn.ub.es/~jmparra

Journal pages

Advances in Applied Clifford Algebras http://www.clifford.org/journals/jadvclfa.html

Subject pages

European Mathematical Information Service http://www.emis.de/
The Geometry Center http://www.geom.umn.edu/
The geometry of the sphere http://math.rice.edu/~pcmi/sphere/
Geometric Calculus: Research and Development http://modelingnts.la.asu.edu/
The Geometric Algebra Research Group (Cambridge) http://www.mrao.cam.ac.uk/~clifford/
Clifford Research Group (Gent) http://cage.rug.ac.be/~fb/crg/

INDEX

A
addition identities, 57
affine plane, 31
affinity, 51
algebra
fundamental theorem, 24
geometric, of the
Euclidean space, 170
pseudo-Euclidean space, 188
algebraic equation of a line, 37
algebraic equations of a circle, 80
alignment of three points, 35
altitudes of a triangle, 73
analytic
continuation, 151
complex functions, 19
hyperbolic functions, 147
analyticity conditions
complex, 19
hyperbolic, 148
angle
between circles, 89
between Euclidean vectors, 12
between hyperbolic vectors, 156
of parallelism, 206, 260 (**15.1**)
inscribed in a circle, 55
antigraphy, 116
Apollonius' theorem
theorem for the ellipse, 128
theorem for the hyperbola, 163
lost theorem, 79, 229 (**8.3**)
area, 33
of a parallelogram, 4
of a triangle, 35, 68
of a spherical triangle, 176
of Lobachevskian triangle, 206, 263 (**15.6**)
on the hyperbolic plane, 161
alignment of three points, 35
analytic functions, 19
angle inscribed in a circle, 55
associative property, 5
azimuthal equivalent projection of the hyperboloid, 200

B
base of vectors for the plane, 10
barycentric coordinates, 33
Beltrami's disk, 191
binomial form of a complex number, 15
bisectors,
of the sides, 70
of the angles, 72
Bretschneider's theorem 91, 238 (**9.4**)
Brianchon's theorem, 138, 248, (**11.8**)

C
Cartesian equation of a circle, 80
Cauchy-Riemann conditions, 19
Cauchy's
theorem, 20
integral formula, 21
central
equation for conics, 126
projection
of the sphere, 177
of the hyperboloid, 191
centroid, 69
Ceva's theorem, 64
cevian lines, 69
Chasles' theorem, 122
circle
of the nine points, 85
inscribed in a triangle, 72, 73
circumscribed to a triangle, 70, 73
circles, 80, 236
radical axis, 89
radical centre, 90
circular functions, 54
circumcentre, 70
circumscribed quadrilateral, 88
Clifford, vi, viii, 8
Clifford algebras
isomorphism of, 152
complex analytic functions, 19
complex numbers, 13
binomial form, 13
polar and trigonometric form, 13
matrix representation of, 139
algebraic operations, 14
complex plane, 18
conformal
cylindrical projection, 202
transformations, 66
congruence
of hyperbolic segments, 158
congruence of Lobachevskian triangles, 205
conic sections, 117
central equations, 126

- in barycentric coordinates, 132
- directrices, 120
- foci, 120
- matrix of, 136
- passing through five points, 131
- perpendicular to
- tangent to, 124
- tangential conic, 132
- vectorial equation, 121

conical projections
- of the sphere, 184
- of the hyperboloid, 203

conjugate
- diameters, 129, 163
- of a complex number, 16
- of a hyperbolic number, 141

convergence
- of complex powers series, 23
- of hyperbolic powers series, 150
- radius of, 22
- square of, 150

coordinate systems, 31
coordinates
- barycentric, 33
- Cartesian, 36
- dual, 43
- homogeneous, 102

correlation, 134
cosines law
- Euclidean, 60, 223 (**6.1**)
- hyperbolic, 164
- hyperboloidal, 198
- spherical, 174, 175

cross ratio
- complex, 92
- projective, 99

cylindrical projections
- of the sphere, 182
- of the hyperboloid, 201

cyclic quadrilaterals, 87

D

De Moivre's identity, 58
Desargues' theorem, 47
diameters
- of the ellipse, 128
- of the hyperbola, 163

dilatations, 30
direct similarity, 61
distance between two points
- on the Euclidean plane, 33
- on the hyperbolic plane, 160

distance from a point to a line, 37
distortion, 177
double angle identities, 55
dual coordinates, 43
dual spherical triangle, 175
duality principle, 43

E

eccentricity, 118
ellipse,
- central equation, 126
- Apollonius' diameters, 128

equations
- of a circle, 80
- of a line, 36-40
- of conics, 119, 121, 126

equivalent cylindrical projection
- of the sphere, 184
- of the hyperboloid, 203

Euler's line, 76
Euclidean space, 170
excess, spherical, 177
exponential
- complex, 14
- hyperbolic, 143

exterior product, 4

F

Fermat's theorem, 77
focal distance
- of a conic, 121
- of a photograph, 186, 258 (**14.4**)

foci of a conic, 120
function of Riemann, 150

G

geometric algebra
- of the vectorial plane, 8
- of the pseudo-Euclidean space, 188

Gibbs, vi, viii
gnomonic projection, 177
Grassmann, vii,viii

H

Hamilton, viii, 8, 171
harmonic
- characteristic, 94
- ranges, 94

Heron's formula, 9, 208 (**1.6**)
hierarchy of algebraic operations, 8
Hilbert, 190
homogeneous coordinates, 102
homography, 96
homologous sides, 61

homology, 110
homothety, 65
horocycle, 206, 263 (**15.4**)
hyperbola
    central equation, 126
hyperbolic
    analyticity, 150
    analytic functions, 147
    cosine, 142
    exponential and logarithm, 143
    numbers, 140
        polar form, 144
        powers and roots, 144
    plane
        distance between points 160
        area on , 161
    sector, 142
    similarity, 167
    sine, 142
    tangent, 142
    trigonometry, 141
    vectors, 154
hyperboloid of two sheets, 190
hyperboloidal
    arc, 193
    defect, 206, 263 (**15.5**)
    triangle, 196
    trigonometry, 196

I
imaginary unity, 13
incentre, 72
infinity, points at, 102
inner product
    of hyperbolic vectors, 155
intercept equation of a line, 40
intersection
    of a line with a circle, 80
    of two lines, 41
inverse
    of a vector, 7
    trigonometric functions, 59
inversions of vectors, 29
inversion with respect to a circle, 83
isometries
    hyperbolic, 158
isomorphism of Clifford algebras, 152
isosceles triangle theorem, 158

L
Lauren series, 22
Leibniz, vi, 190
line at the infinity, 47
lines on the Euclidean plane, 31
linear combination of two vectors, 10
Liouville's theorem, 24
Lobachevskian
    geometry, 188
    trigonometry, 196
logarithm
    complex, 20
    hyperbolic, 143

M
McLaurin series, 22
matrix representation, 139
medians, 69
Menelaus' theorem, 63
Mercator's projection, 183
Möbius, viii, 33, 78
    transformation, 96
modulus
    of a complex number, 13
    of a hyperbolic number, 141
Mollweide's formulas, 60

N
Napier's rule, 176
Napoleon's theorem, 78
Newton's theorem, 138
nine-point circle, 85
non-Euclidean geometry, 205

O
opposite similarity, 62
orthocentre, 73
orthogonal hyperbolic vectors, 157
orthographic projection of the sphere, 181
orthonormal bases, 11
outer product
    of Euclidean vectors, 4
    of hyperbolic vectors, 155

P
parametric equations of a line, 36
Pascal's theorem, 138, 248, (**11.7**)
Pauli's matrices, 153
Peano, vi, vii, 128
pencil of lines, 41
permutation of complex and vector, 17
permutative property
    of Euclidean vectors, 6, 9, 207 (**1.5**)
    of hyperbolic vectors, 155
perspectivity, 103
Peter's projection, 184
Plate Carré projection, 182
Poincaré's

disk, 198
half-plane, 200
points, 31
alignment, 31
at the infinity, 102
on the Euclidean plane, 214
polar equation of a line, 40
polar equation of a circle, 82
polar form of a complex number, 14
polarities, 134
poles, 23
polygon
sum of the angles, 53
power of a point
with respect to a circle, 82
with respect to a hyperbola, 168
product
of a vector and a real number, 2
of two vectors, 2
of three vectors, 5
of four vectors, 7
inner
of hyperbolic vectors, 155
outer
of hyperbolic vectors, 155
projection
of the spherical surface
central or gnomonic, 177
conical, 184
Mercator's, 183
Peter's projection, 184
Plate Carré's, 182
orthographic, 181
stereographic, 180
of the hyperboloid
central (Beltrami's), 191
conical, 203
stereographic, 198
azimuthal equivalent, 200
cylindrical equidistant, 201
cylindrical conformal, 202
cylindrical equivalent, 203
projective cross ratio, 99
projectivity, 103
pseudo-Euclidean
plane, 154
space, 188
Ptomely's theorem, 115, 240, (**10.1**)

Q
quadrilaterals
cyclic, 87
circumscribed, 88
quaternions, 8, 171
quotient
of two vectors, 7
of complex numbers, 16

R
radical axis of two circles, 89
radical centre of three circles, 90
ratio
complex cross, 92
of similarity, 61
projective cross, 99
single, 65
reflections of vectors, 28
residue theorem, 24
reversal, 168
Riemann's function, 150
rotations of vectors, 27, 139

S
sector of a hyperbola, 142
series
Lauren, 22
McLaurin, 22
Taylor, 21
similarity
direct, 61
opposite, 62
hyperbolic, 167
Simson's theorem, 91, 237, (**9.3**)
sines law
Euclidean, 60, 223 (**6.1**)
hyperbolic, 164
hyperboloidal, 197
spherical, 173
single ratio, 65
singularities, 23
slope-intercept equation of a line, 40
space
Euclidean, 170
pseudo-Euclidean, 188
spherical triangle
excess of, 177
dual, 175
area of, 176
spherical trigonometry, 172
spherical coordinates, 182
stereographic projection
of the sphere, 180
of the hyperboloid, 198
strict triangles, 173
sum of the angles
of a polygon, 53

of a hyperbolic triangle, 160
sum of trigonometric functions, 56

T
tangents law 60, 169, 254 (**13.5**)
tangent to a conic, 124
Taylor series, 22
tetranions, 188
theorem
of the algebra, fundamental, 24
of Apollonius, 128, 138, 163, (**11.3**)
lost, 78, 229, (**8.3**)
of Brianchon, 138, 248, (**11.8**)
of Cauchy, 20
of Ceva, 64
of Chasles, 122
of Desargues, 47
of Fermat, 77
of Liouville, 24
of Menelaus, 63
of Napoleon, 78
of Newton, 138
of Pascal, 138, 248, (**11.7**)
of Ptolemy, 115, 240, (**10.1**)
of Simson, 91, 237, (**9.3**)
of the residue, 24
of the isosceles triangle, 158
theorems
projective demonstration, 108
about hyperbolic angles, 160
transformations
of vectors, 213
special conformal, 115, 242, (**10.5**)
translations, 31
triangle
dual spherical, 175
triangular inequality, 34
trigonometric
addition identities, 57
double angle identities, 55
form of a complex number, 14
functions, 54
fundamental identities, 54
inverse functions, 59
sum of functions, 56
product of vectors
trigonometry
circular, 53
hyperbolic, 141
hyperboloidal (Lobachevskian), 196
spherical, 172

V
vector
addition, 1
product, 2
transformations, 213
vectorial equation of a line, 36

W
Weierstrass'
coordinates, 201
model, 190

CHRONOLOGY OF THE GEOMETRIC ALGEBRA

1679 Letters of Leibniz to Huygens on the *characteristica geometrica*.
1799 Publication of *Om Directionens analytiske Betegning* by Caspar Wessel with scarce diffusion.
1805 Birth of William Rowan Hamilton at Dublin.
1806 Publication of *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* by Jean Robert Argand.
1809 Birth of Hermann Günther Grassmann at Stettin.
1818 Death of Wessel
1822 Death of Argand.
1827 Publication of *Der barycentrische Calcul* by Möbius at Leipzig.
1831 Birth of James Clerk Maxwell at Edinburgh.
1831 Birth of Peter Guthrie Tait.
1839 Birth of Josiah Willard Gibbs at New Haven.
1843 Discovery of the quaternions by Hamilton.
1844 Publication of *Die Lineale Ausdehnungslehre* (first edition), where Grassmann presents the anticommutative product of geometric unities (outer product).
1845 Birth of William Kingdon Clifford at Exeter.
1847 Publication of *Geometric Analysis* with a foreword by Möbius, memoir with which Grassmann won the prize to whom developed the Leibniz's *characteristica geometrica*.
1850 Birth of Oliver Heaviside at London.
1853 Publication of *Lectures on Quaternions* where Hamilton introduces the nabla operator (gradient).
1862 Publication of the second edition of *Die Ausdehnungslehre.*
1864 Publication of *A dynamical theory of the electromagnetic field* by Maxwell, where he defines the divergence and the curl.
1865 Death of Hamilton.
1866 Posthumous publication of Hamilton's *Elements of Quaternions*.
1867 Publication of *Elementary Treatise on Quaternions* by Tait.
1873 Publication of *Introduction to Quaternions* by Kelland and Tait.
1873 Maxwell publishes the *Treatise on Electricity and Magnetism* where he writes the equations of electromagnetism with quaternions.
1877 Publication of the Grassmann's paper «Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre».
1877 Death of Grassmann.
1878 Publication of the paper «Applications of Grassmann's Extensive Algebra» by Clifford where he makes the synthesis of the systems of Grassmann and Hamilton.
1879 Death of Maxwell.
1879 Death of Clifford.
1880 Publication of Lipschitz' *Principes d'un calcul algebraic*.
1881 Private printing of *Elements of Vector Analysis* by Gibbs.
1886 Publication of Lipschitz' *Untersuchungen uber die Summen von Quadraten*.
1886 Publication of the Gibbs' paper «On multiple algebra».
1888 Publication of Peano's *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazione della logica deduttiva*.
1891 Oliver Heaviside publishes «The elements of vectorial algebra and analysis» in *The Electrician Series*.
1895 Publication of Peano's «Saggio di Calcolo Geometrico».
1901 Death of Tait.
1901 Wilson publishes the Gibbs' lessons in *Vector Analysis*.
1903 Death of Gibbs.

1925 Death of Heaviside.
1926 Wolfgang Pauli introduces his matrices to explain the electronic spin.
1928 Publication of the paper «The Quantum Theory of Electron», where Paul A. M. Dirac defines a set of 4×4 anticommutative matrices built from the Pauli's matrices.

This comparative diagram of the life and works of the authors of (or related with) the geometric algebra visualises and summarises the chronology. The XIX century may be properly called *the century of the geometric algebra*. Note the premature death of Clifford, which caused the delay in the development of the geometric algebra along the XX century.

```
                 1800      1820      1840      1860      1880      1900
                   |         |         |         |         |         |

Wessel**************
Argand*************************
Hamilton              *******************************
Grassmann               ***********************************
Maxwell                            *************************
Tait                               ************************************
Gibbs                                  *********************************
Clifford                                  ******************
Heaviside                                   **************************************


Essai sur ...         |
Der barycentrische Calcul        |
Die Ausdehnungslehre                     |
Lectures on Quaternions                       |
Elements on Quaternions                             |
Elementary Treatise on Quaternions                   |
Treatise on Electricity and Magnetism                   |
Applications of Grassmann's Extensive Algebra             |
Elements of Vector Analysis                                 |
The Electrician Series                                           |
Vector Analysis                                                        |
```