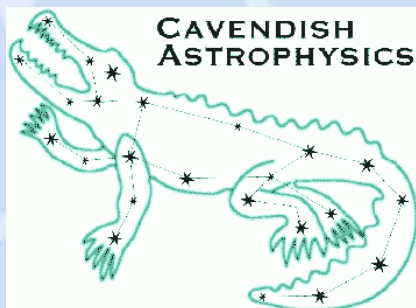
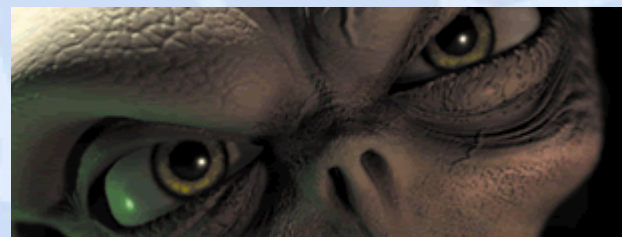


Modeling the Cosmos: The Shape of the Universe

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Overview

Want to share two recent exciting developments

- Recent progress in cosmology
- Recent progress in geometrical description
 - Applicable in computer graphics and robotics

Cosmology:

- May be close now to understanding the geometry of the Universe
- Pretty sure now about its age and fate
 - About 14 billion years old + expanding forever at an accelerating rate!

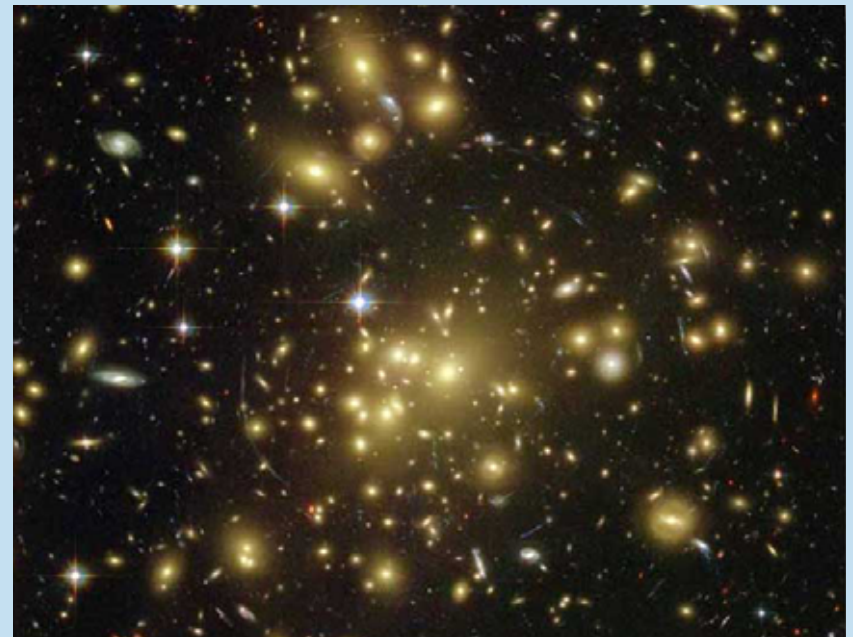
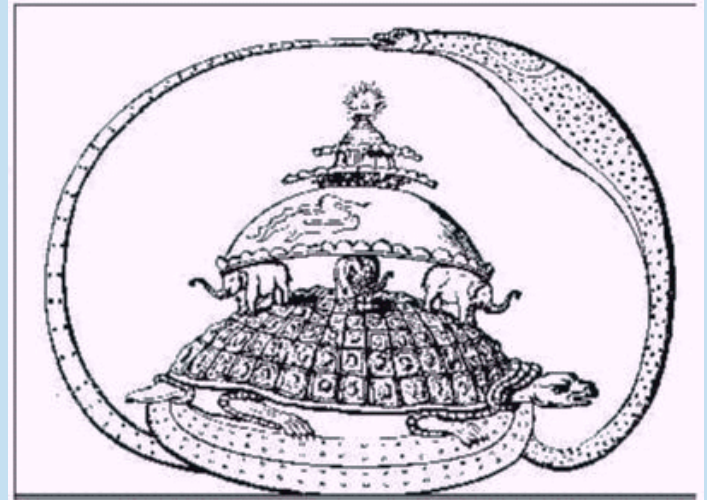
Overview



SIGGRAPH 2003
SAN DIEGO

- Basically at last getting quantitative answers to some of the oldest questions humanity has asked
- But while quantitative, not sure exactly what we are measuring: the universe seems to consist of

5% ordinary matter
25% "dark matter"
70% "dark energy" } What are these?



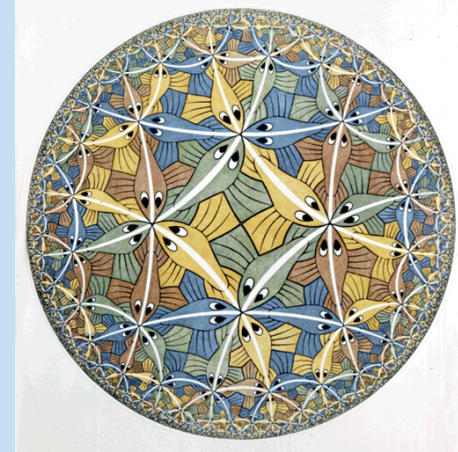
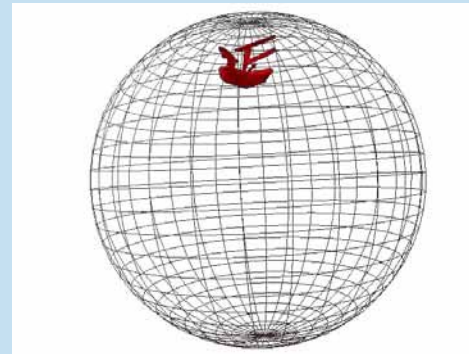
Overview - geometry

- Recent exciting advances in geometrical description
- A unifying language now possible which encompasses all of:

- Euclidean
- Hyperbolic
- Spherical
- Projective
- Affine

geometries in a simple way

- Links through seamlessly with many other areas of maths, physics and engineering (including computer graphics)



- Can easily do 3d version of 2d 'Poincare disc' (e.g. as in Escher)
- Above shows starship in a 3d hyperbolic space
- Call the new technique 'conformal geometric algebra'

(Mathematics)



Note: For those of you not too used to working with equations, or are not sure what the above geometries are: Don't worry!

- Will be some equations, but in general can ignore them, and overall flow should be same
- Also, one of the points of the new geometrical approach is that can start to do geometry by stringing together:

“words” $\leftrightarrow$ “geometrical objects”

“sentences” $\leftrightarrow$ “relations between the objects”

in a new intuitive way that everyone can carry out and appreciate

- This has implications for computing and graphics – conceptually much easier to do geometry (even if computing speed similar)

Mathematics and the two themes

Thus the 'conformal geometric algebra' provides a genuine new language (and will explain some features of above geometries in this context)

How do the two themes link?

The geometrical description applies in any dimension and even in 4-dimensional spacetime

- We're going to do some geometry in that space!



E.g. here is the starship moving in **de Sitter** space – constant curvature spacetime

- Very important in cosmology
- We'll see how easy it is to make the transition to this from the space of ordinary life (Euclidean 3-space)
- Again, starts to make these things accessible to **everybody**

The Universe

Find if ask that this new description applies to the Universe, then implies physical restrictions

- In particular that the Universe is “**closed**” (will explain)
- Predicts the “**dark energy**” and roughly its magnitude – geometrically!

Particle physicists try to do this, but (they won't mind me saying) they get it wrong by a factor 10^{122} !

So let's make a start on each theme

Geometric Algebra



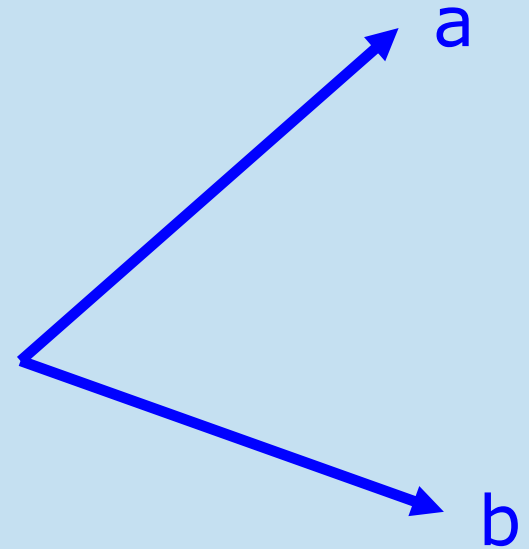
If know about complex numbers, then know there is a 'unit imaginary' i

- Main property is that $i^2 = -1$
- How can this be? (any ordinary number squared is positive)
- Troubled some very good mathematicians for many years
- Usually these days an object with these properties just defined to exist, and 'complex numbers' are defined as
 $x + i y$ (x and y ordinary numbers)

Geometric Algebra-II

But consider following:

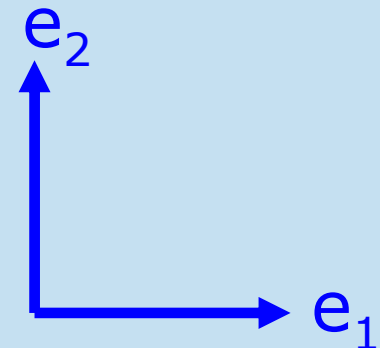
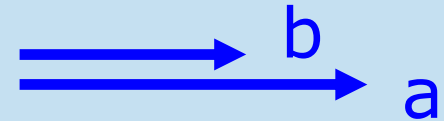
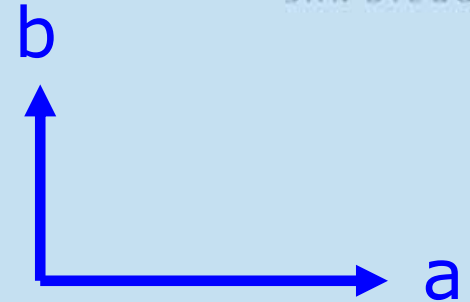
- Suppose have two directions in space **a** and **b** (these are called 'vectors')
- And suppose we had a language in which we could use vectors as words and string together meaningful phrases and sentences with them
- So e.g. **ab** or **bab** or **abab** would be meaningful phrases



Geometric Algebra-III

Now introduce two rules:

- If a and b perpendicular, then $ab = -ba$
- If a and b parallel (same sense) then $ab = |a||b|$ (product of lengths)
- Just this does an amazing amount of mathematics!
- E.g. suppose have two unit vectors at right angles
- Rules say $e_1^2 = e_1e_1 = 1$, $e_2^2 = e_2e_2 = 1$ and $e_1e_2 = -e_2e_1$



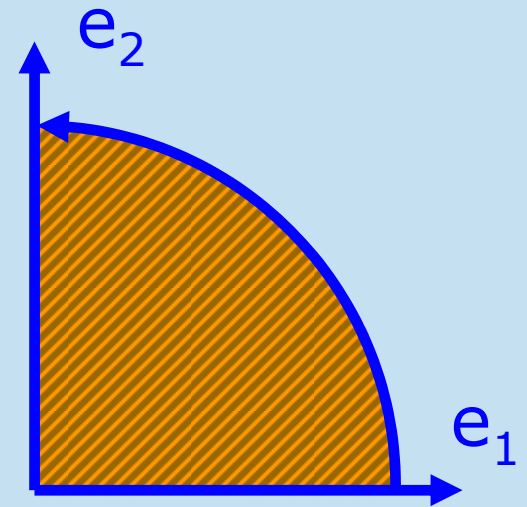
Geometric Algebra-IV

Try $(e_1 e_2)^2$

- This is

$$e_1 e_2 e_1 e_2 = - e_1 e_1 e_2 e_2 = -1$$

- We have found a geometrical object $(e_1 e_2)$ which squares to minus 1 !
- Can now see complex numbers are objects of the form $x + (e_1 e_2) y$
- What is $(e_1 e_2)$? – we call it a **bivector**
- Can think of it as an oriented plane segment swept out in going from e_1 to e_2



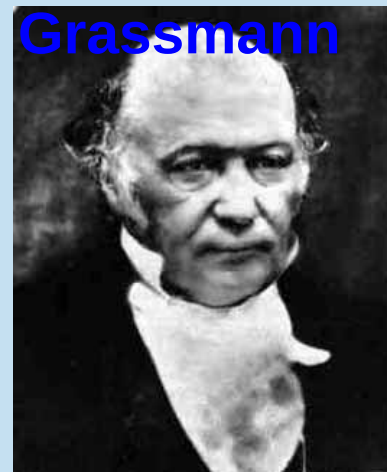
Development of Geometric Algebra

These sort of structures introduced by Grassmann and Clifford

- **Grassmann** (1809-1877) was a German schoolteacher
- Disappointed in lack of interest in his mathematical ideas – turned to Sanskrit (dictionary still used)
- **Clifford** (1845-1879) Cambridge mathematician and philosopher
- United Grassmann's ideas with the quaternions of **Hamilton**



**Hermann
Grassmann**



William



**William
Clifford**

GA as a language

- Turning GA into a general tool, applicable to a great deal of maths and physics, carried out by **David Hestenes** (Oersted medal winner)
- Pursuing idea of a language, how do objects like $x + (e_1 e_2) y$ fit in?
- Note it is not itself a vector
- Removing an overall scale factor, we call it a **rotor** R
- (If leave the scale factor in, called a 'spinor' – some will know this from quantum mechanics)
- Their key role is to rotate things!

The language of rotations



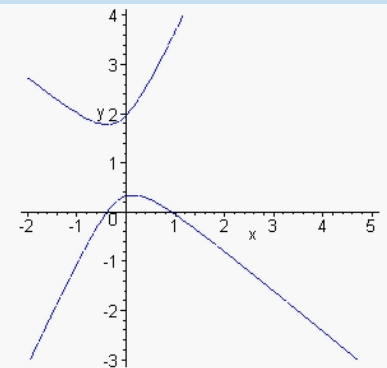
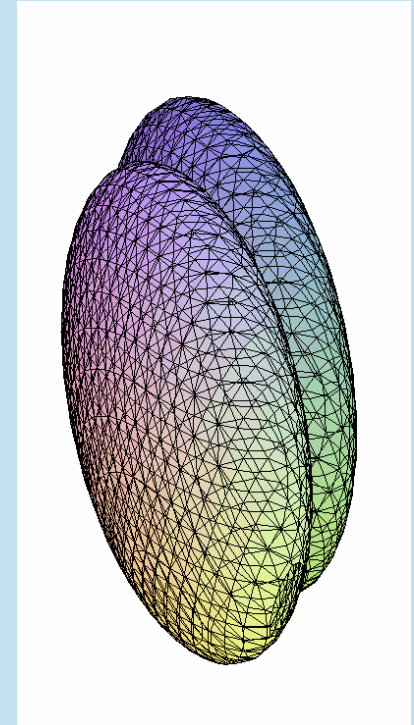
- Appropriate R 's exist in any dimension, and even in relativistic spaces
- E.g. in 3d the R 's are quaternions
in 4d spacetime they carry out Lorentz transformations
- Won't discuss the details of how it works, but the rotors allow the rotated objects still to be combined together in the language
- All combinations still valid

Translations?

So can rotate things easily, and have a language involving the rotated objects

Now, here is the huge step the CGA achieves for us

- It enables **translations** (rigid displacements from one position to another) to be represented by rotors
- Works in a space 2d up from the base space
- E.g. Euclidean 3-space needs 5d
Spacetime (3 space, 1 time) needs 6d
- Seems wasteful, but: doing translations with rotors means they are integrated into the 'language'
- Turns out "objects" can include all of **spheres, ellipsoids, hyperboloids, and circles**, as well as **planes and lines**



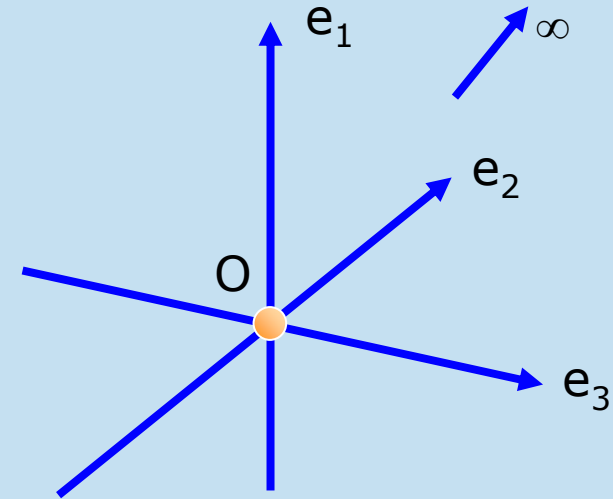
The Conformal GA

How it works, is that we adjoin two extra vectors to our space:

e squares to $+1$

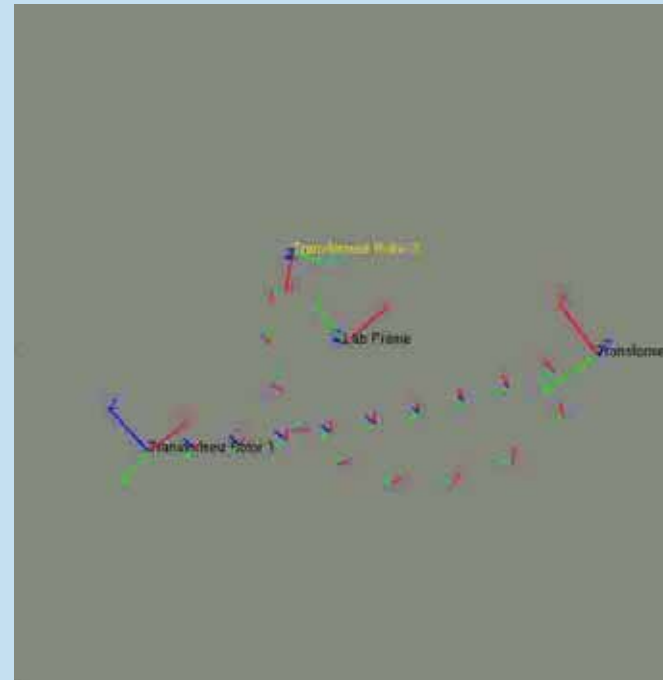
$\bar{e}$ squares to -1

- Vector x labelling position in 3d is associated with a null vector X in 5d (null means $X^2=0$)
- Two special points worth indicating explicitly:
- Origin $x = 0$ is represented by $X = \bar{e} - e$ (check null)
- Point at infinity by $X = \bar{e} + e$
 $= n$ say



Conformal GA contd.

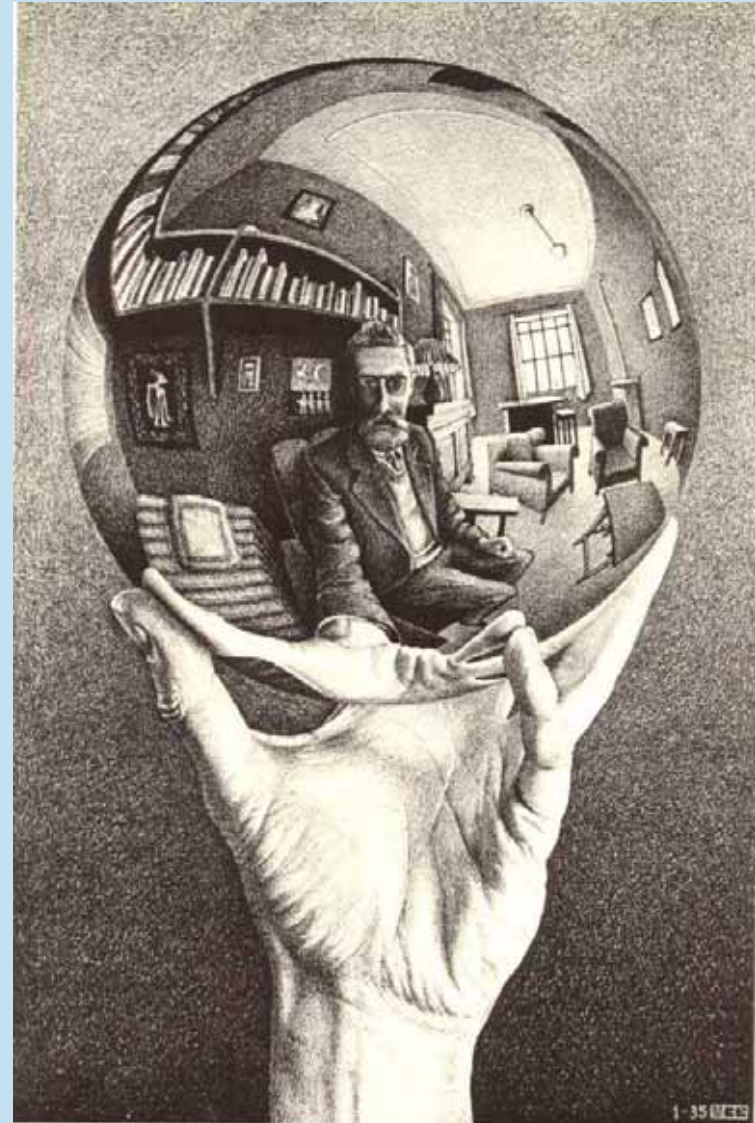
- Do translations in 3 space via rotations in 5 space with a special R
- Now any finite translation can't affect points at infinity
- Whole of Euclidean geometry basically amounts to saying that we use rotors which leave $n = \bar{e} + e$ invariant
- (At least up to scale – turns out dilations are done with a rotor which changes its scale)
- Having things done with rotors is very important e.g. for interpolation:



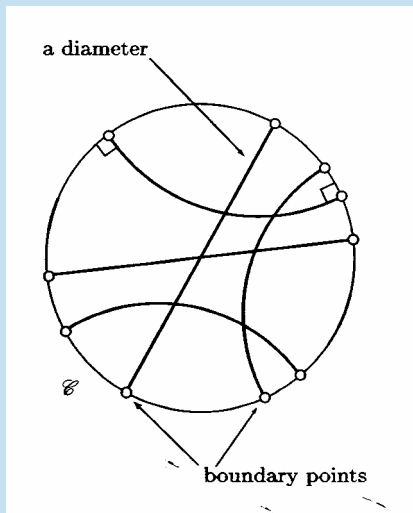
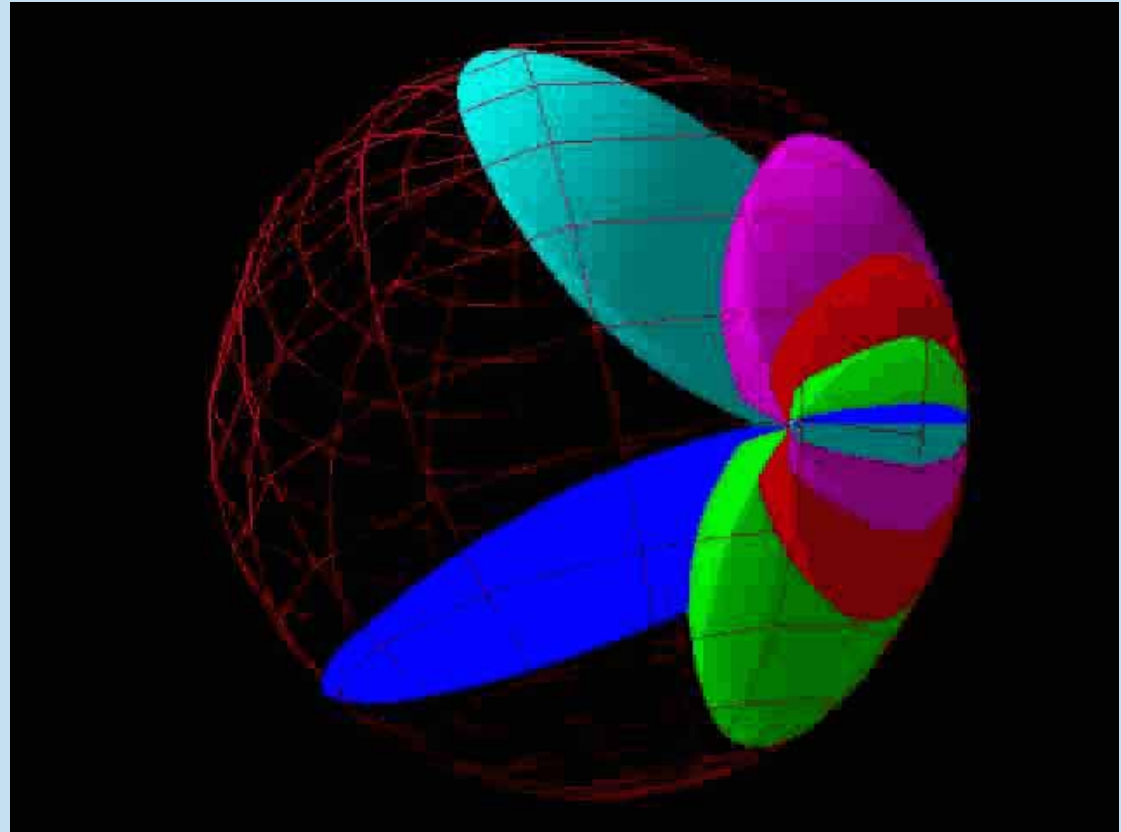
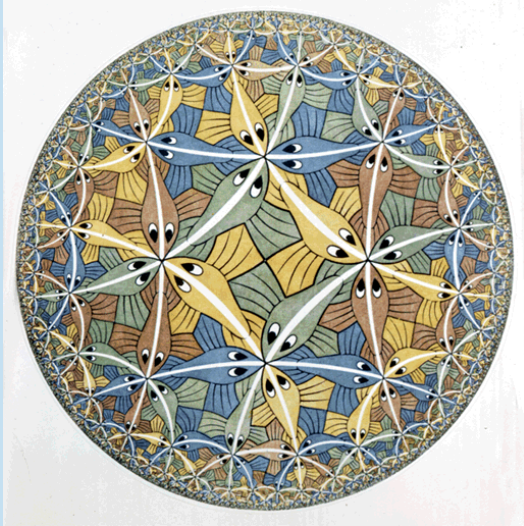
Can interpolate properly between the rotors in the 5d space: implies properly linked interpolation of rotation and translation

Other geometries

- We said Euclidean geometry amounts to rotors which leave $\mathbf{n}$ invariant.
- What if we choose the rotors so as to leave other vectors invariant?
- Find: Look for transformations that keep $\mathbf{e}$ invariant in our 5d space:
⇒ **hyperbolic geometry**
- Look for transformations that keep $\bar{\mathbf{e}}$ invariant in our 5d space:
⇒ **spherical geometry**
- All the structure of the rotor language (interpolation etc.) still available for these cases



Illustrations of Hyperbolic Geometry



Planes in 3d hyperbolic space

Final concepts

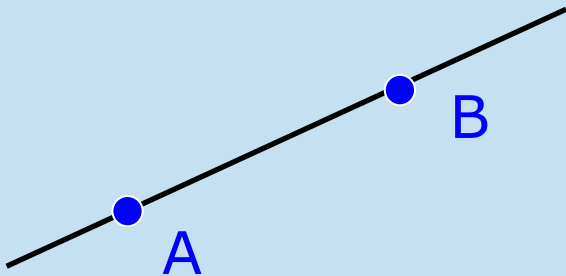
Grades of objects:

- Scalars grade 0
- Vectors grade 1
- Bivectors grade 2
- Trivectors grade 3 ...

Wedge product:

- $A \wedge B$ = bivector part of AB
- $A \wedge B \wedge C$ = trivector part of ABC etc.

Can now do everything we want: e.g. lines are represented by:

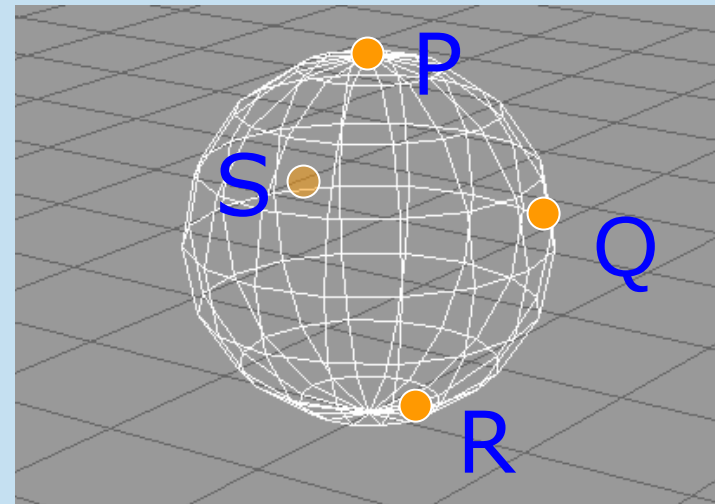
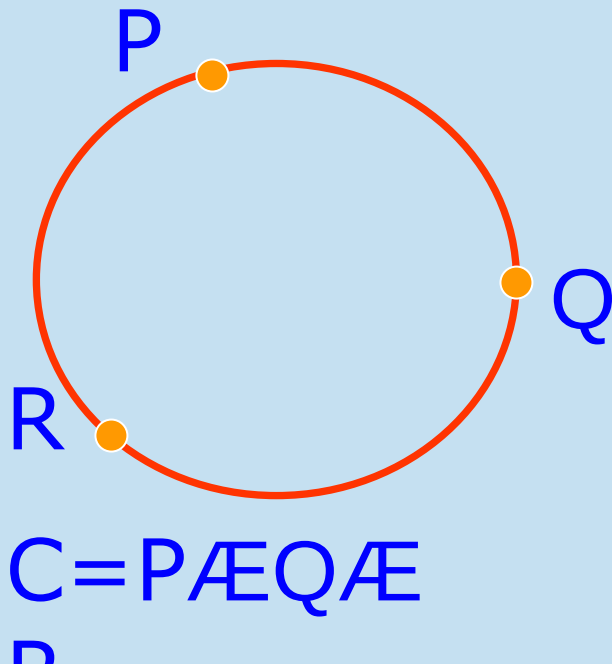
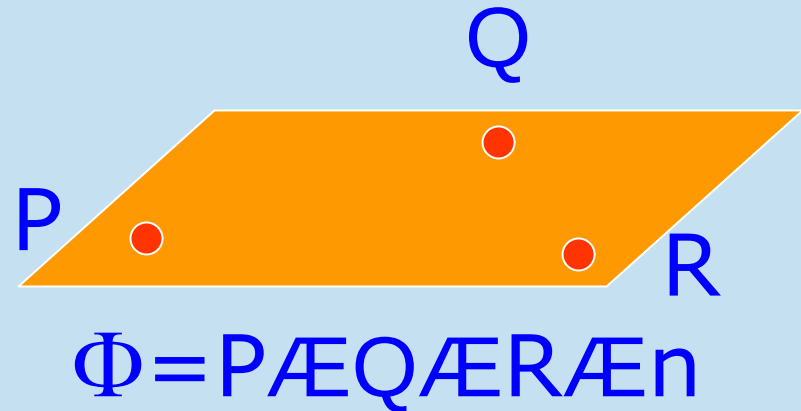
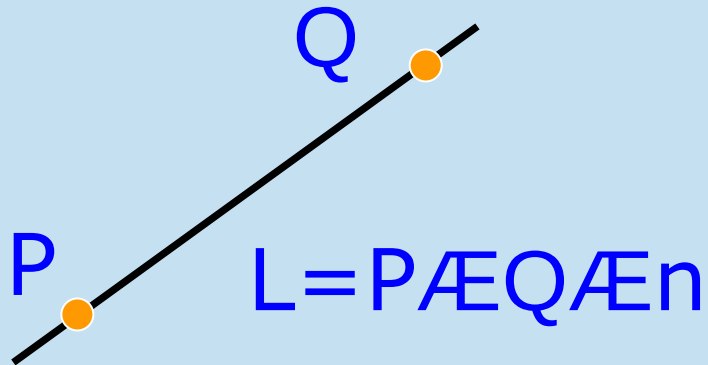


$A \wedge B \wedge n$ Euclidean case

$A \wedge B \wedge e$ Hyperbolic case

$A \wedge B \wedge \bar{e}$ Spherical case

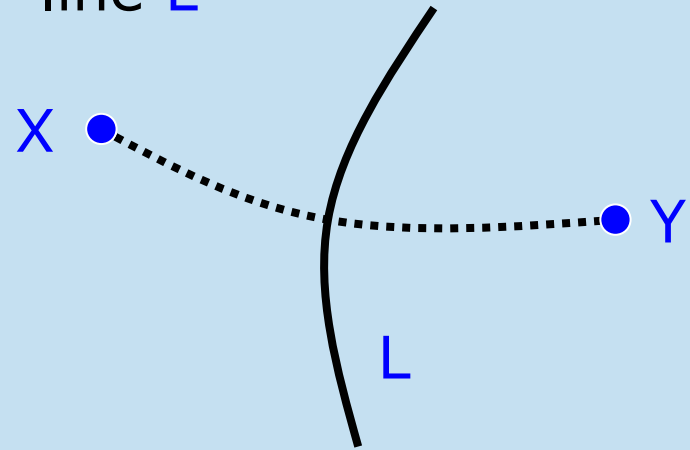
Lines, circles, planes and spheres



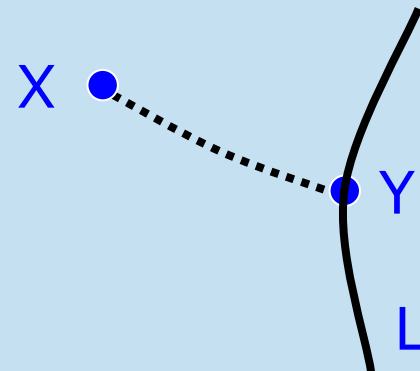
Carrying on

- Can use these objects in our language
- All valid sentences are meaningful
- In each of Euclidean, hyperbolic, spherical space and relativistic versions of each of these
- An amazing unification!
- Some random examples (illustrate here in non-Euclidean hyperbolic plane)

- $Y = L X L$ \$ reflect X in the line L



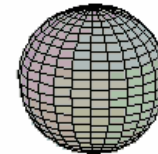
- $Y = X + L X L$ \$ drop a perpendicular to the line L



More examples of the language

Say have two spheres, Σ_1 and Σ_2 and a plane Φ

- $\Sigma_1 \Sigma_2$ is rotor which takes Σ_1 to its reflection in Σ_2
- $1 + \theta \Sigma_1 \Phi$ is rotor which interpolates from Σ_1 to Φ
- $\Sigma_1 \Sigma_2 - \Sigma_2 \Sigma_1$ is circle of intersection of the spheres!
- Etc. Fascinating rich world opens up
- Same methods, tools, results etc. can be applied in any of the spaces



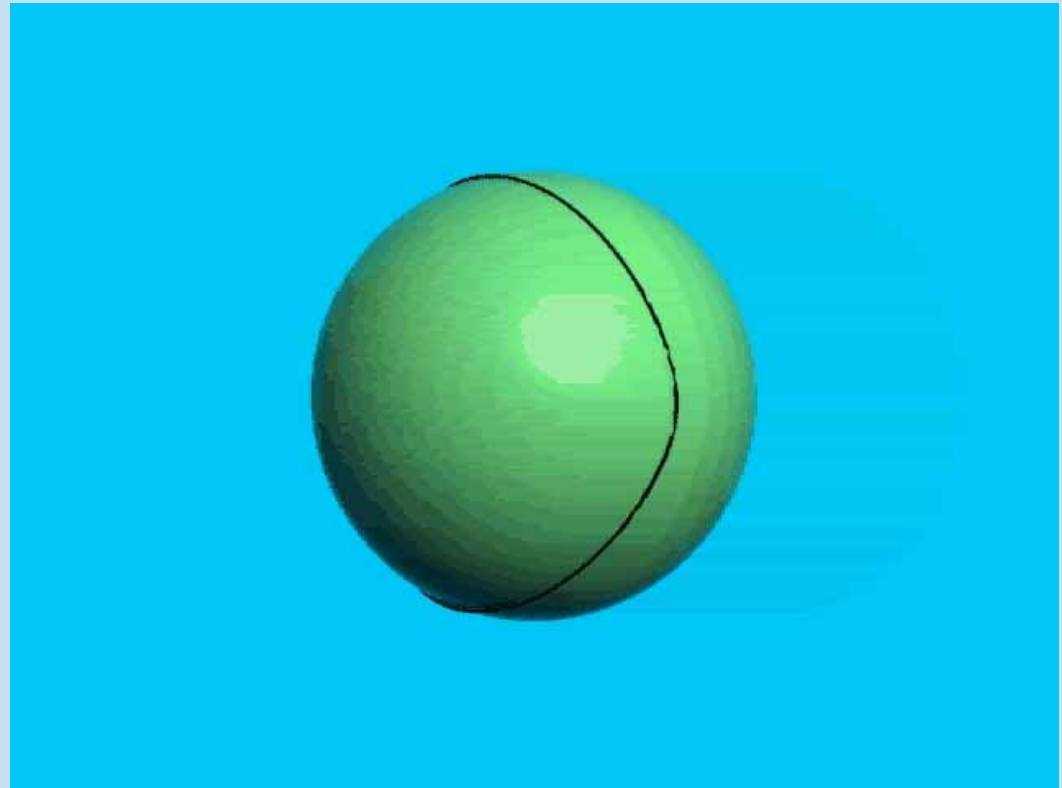
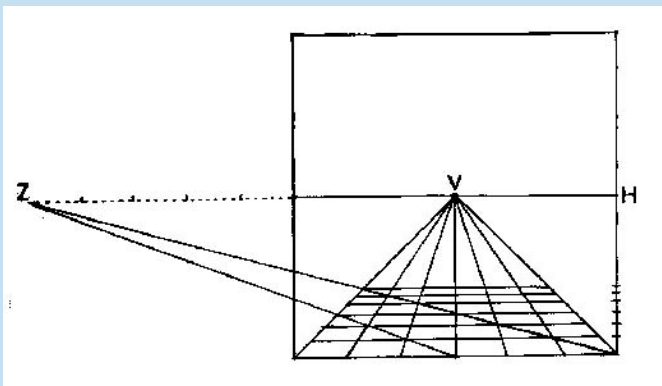
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intersected (everything with everything) in real time – very simple to program

Useful in collision detection etc.

New Geometries

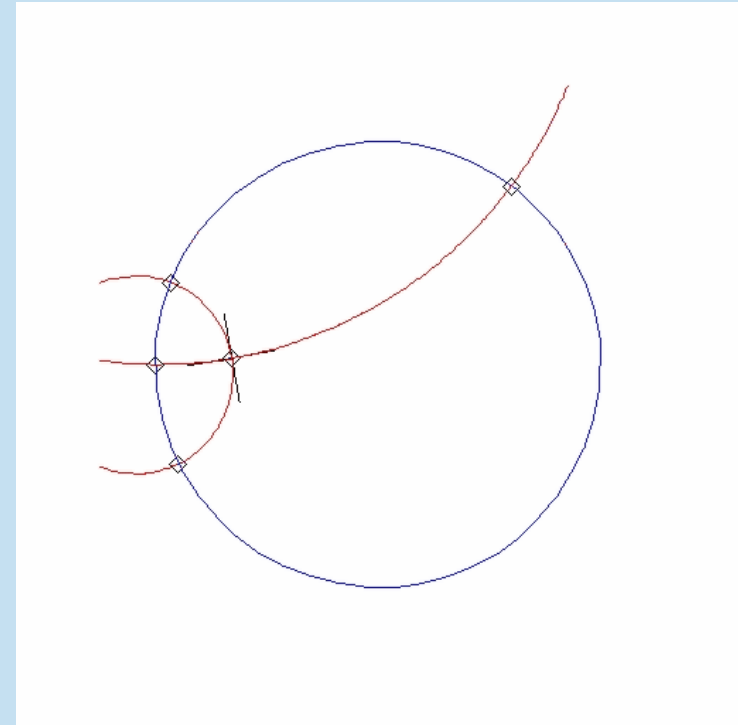
Can even generate new geometries by combining perspective transformations with the non-Euclidean geometry

- Still all done using the null vector approach
- Appears to be new!
- Movie shows a spherical ellipse/hyperbola



The significance of the boundary

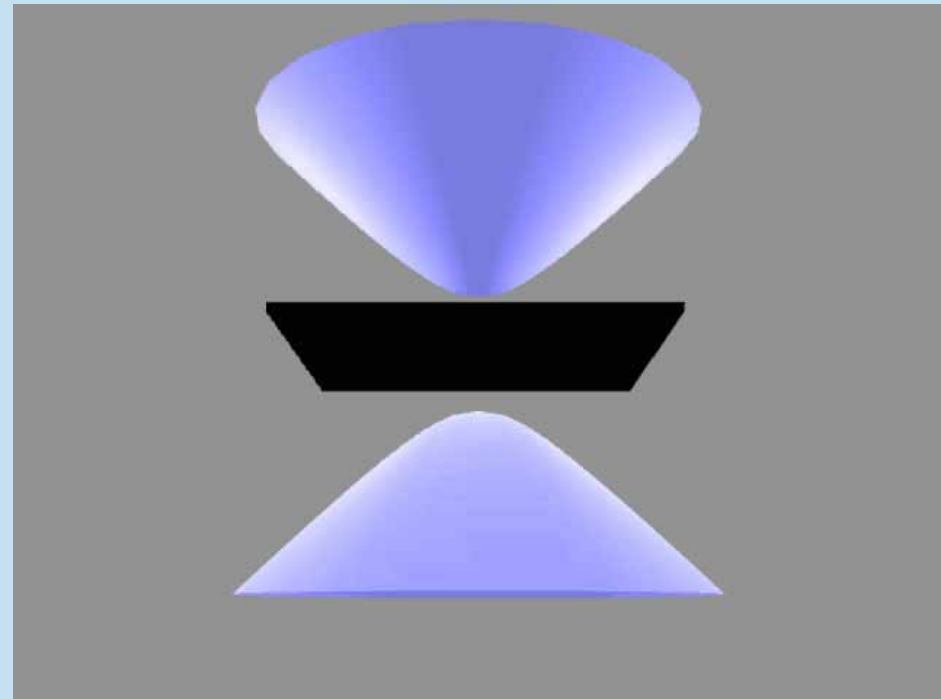
- The boundary to the space seems to have deep connections to both the physics and geometry
- We still do not understand this fully yet
- Here is a nice example – what are 'free' versus 'position' vectors in a hyperbolic space?
- In standard differential geometry, this leads to concepts of 'tangent space' etc - but quite abstract
- Here can give a very direct interpretation



- Key is to trace along the 'geodesics' to the boundary

De Sitter space

- de Sitter space is spacetime $(3+1)$ in which we preserve e
- (Anti de Sitter – very popular with theoretical physicists – we preserve $\bar{e}$)
- Animation shows its boundary plus $t=0$ plane
- Our universe seems to be heading towards de Sitter – does our conformal description have implications for this?

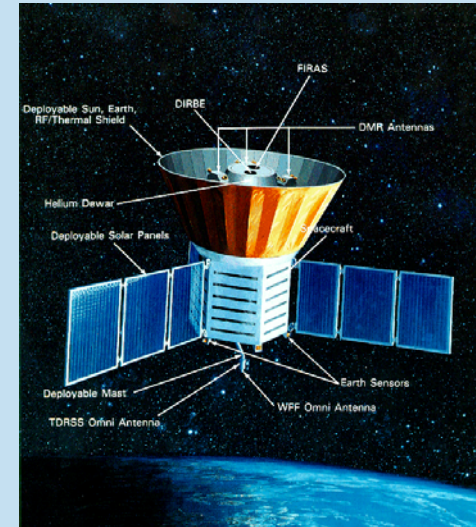


Cosmology

- A key question is: What is the origin of structure?

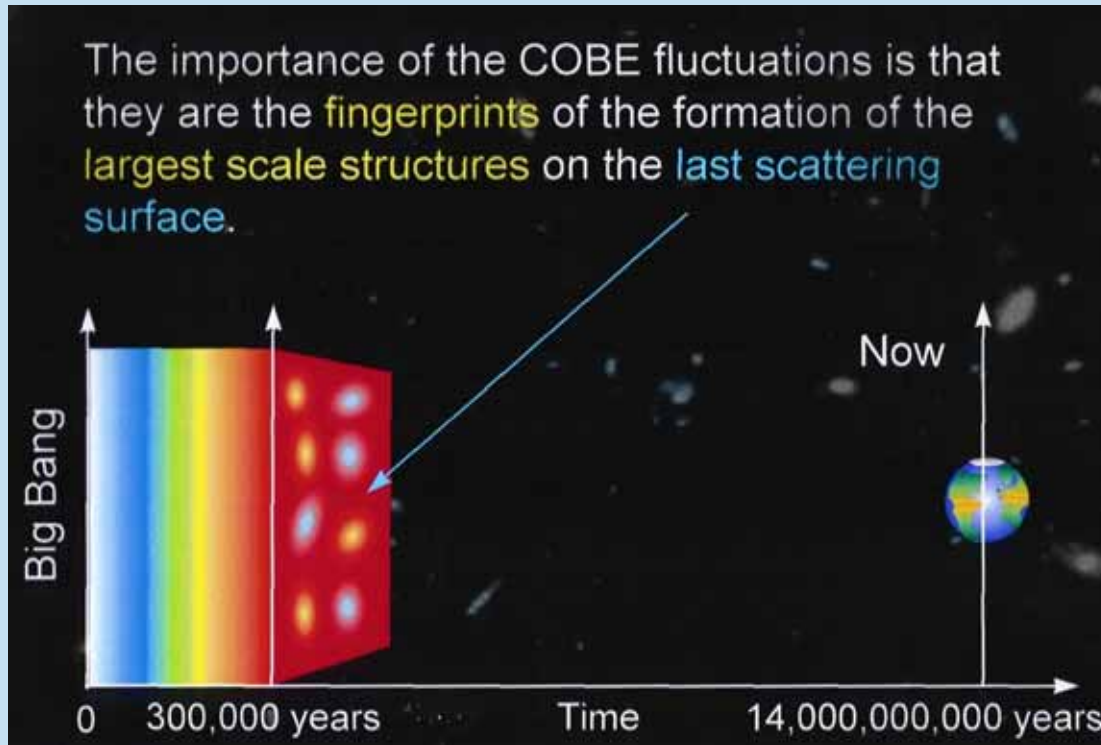
- By this we mean: galaxies, clusters of galaxies exist today – where did they come from – what were the ‘seeds’ from which they developed?

- Key clue to this comes from the ‘Cosmic Microwave Background’
- Discovered by Penzias and Wilson in 1965



- Bath of radiation at 2.7 Kelvin enveloping Earth – extremely uniform in temperature as function of direction
- But not quite! Variations in temperature around 1 part in 10^5 discovered by COBE satellite

CMB fluctuations and structure



The CMB fluctuations relate to 300,000 years after the big bang

What should their matter equivalents have grown into today?



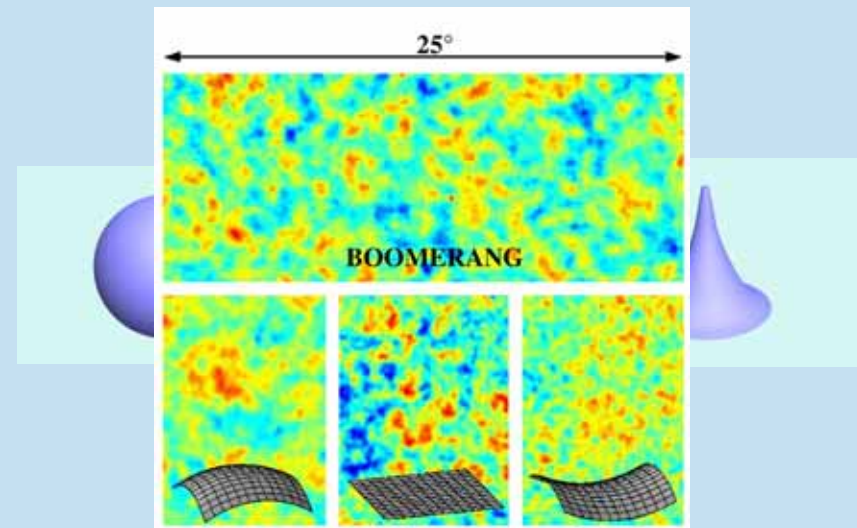
$t=0.0036$



The geometry of the Universe

- Crucial information from each of these is the amplitude of fluctuation as a function of scale (the '**Power Spectrum**')
- E.g. the CMB power spectrum has encoded in it the geometry of the universe:
- The picture shows the typical sky appearance for different types of universe geometry - **closed**, **flat** and **open** -with actual CMB results at the top

Results from a balloon-borne experiment: Boomerang



Left: Universe closed – spatial geometry like a sphere

Middle: Universe flat – geometry just that of Euclidean 3 space

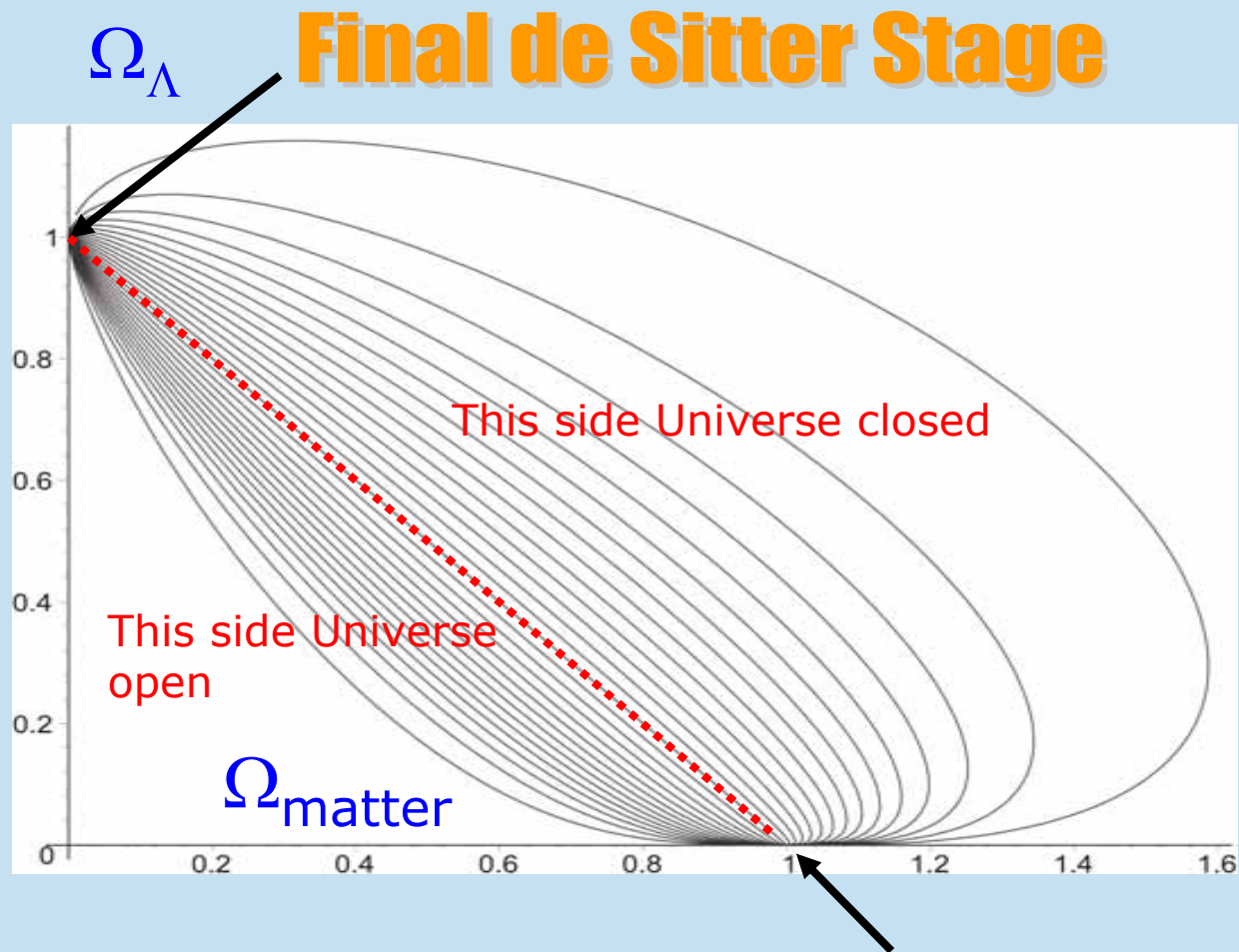
Right: Universe open – geometry hyperbolic

The density and destiny of the Universe



- The three possibilities for geometry correspond to three possibilities for total density: $\Omega = \rho_{\text{actual}} / \rho_{\text{for flat}} = \Omega_{\Lambda} + \Omega_{\text{matter}}$
- Here Λ is the cosmological constant (dark energy)
- Closed: $\Omega > 1$
- Open: $\Omega < 1$
- Flat: $\Omega = 1$
- Usually said that:
 - Closed universe will eventually recontract ($\mapsto$ **Big Crunch**)
 - Flat universe expands forever, and has 0 velocity at infinite time
 - Open universe expands forever, and has positive velocity at infinite time
- With Λ present, dynamics is very different from what people used to think:

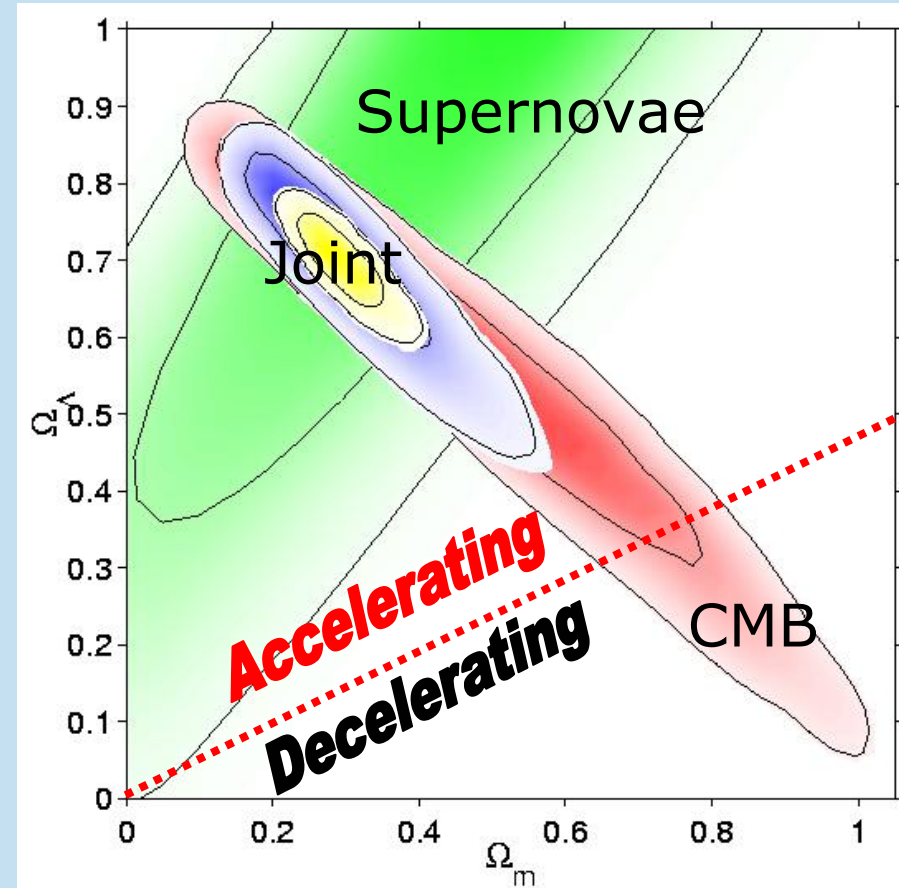
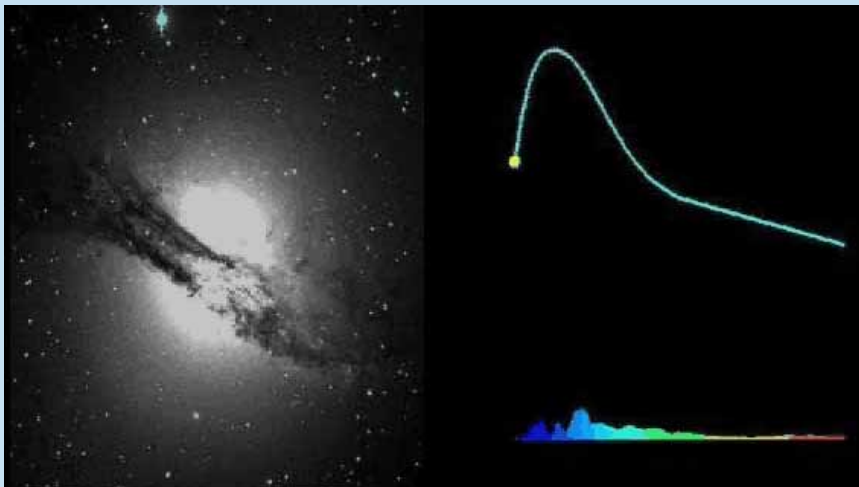
Flow lines for the Universe



Universe starts at $(\Omega_{\text{matter}}, \Omega_\Lambda) = (1, 0)$
and moves to attractor point at $(0, 1)$
(de Sitter) – which curve are we on??

Which flow line?

- Current evidence from the CMB and LSS is that $\Omega_\Lambda \gg 0.7$ and $\Omega_{\text{matter}} \gg 0.3$ – close to flat, but not sure!
- Independent evidence from Supernovae at large distances from us

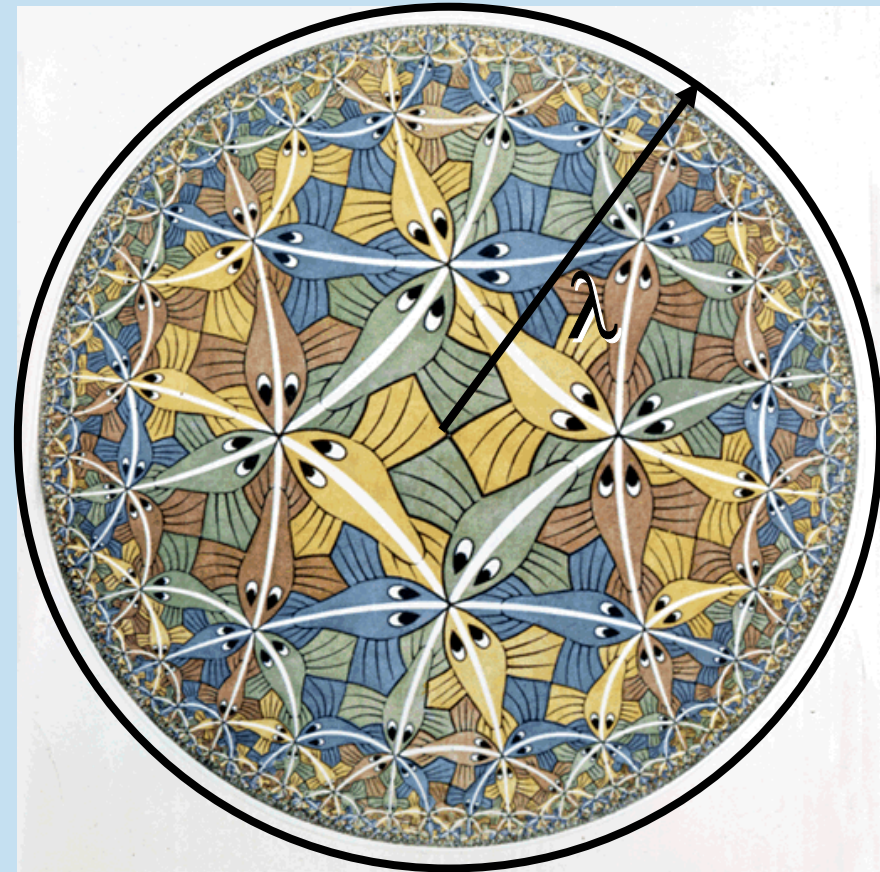


The supernovae are fainter than they should be given their redshifts – indicates the universe is accelerating!

What is Λ ?

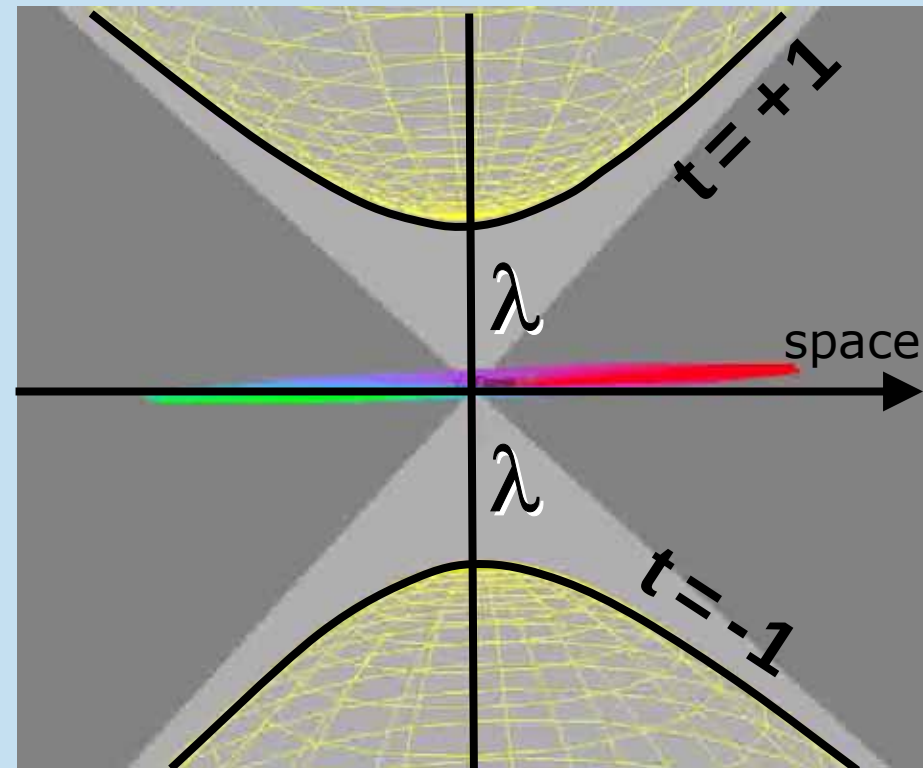
So we are heading towards a de Sitter phase in which Λ dominates

- What is Λ ?
- Normally thought of in terms of particle physics, but then completely unable to explain magnitude (prediction 10^{122} too big)
- In fact, could it be just geometry?
- E.g. the CGA representation of hyperbolic space has a boundary
- Say this boundary at radius λ , then there is an effective cosmological constant in the space / $1/\lambda^2$



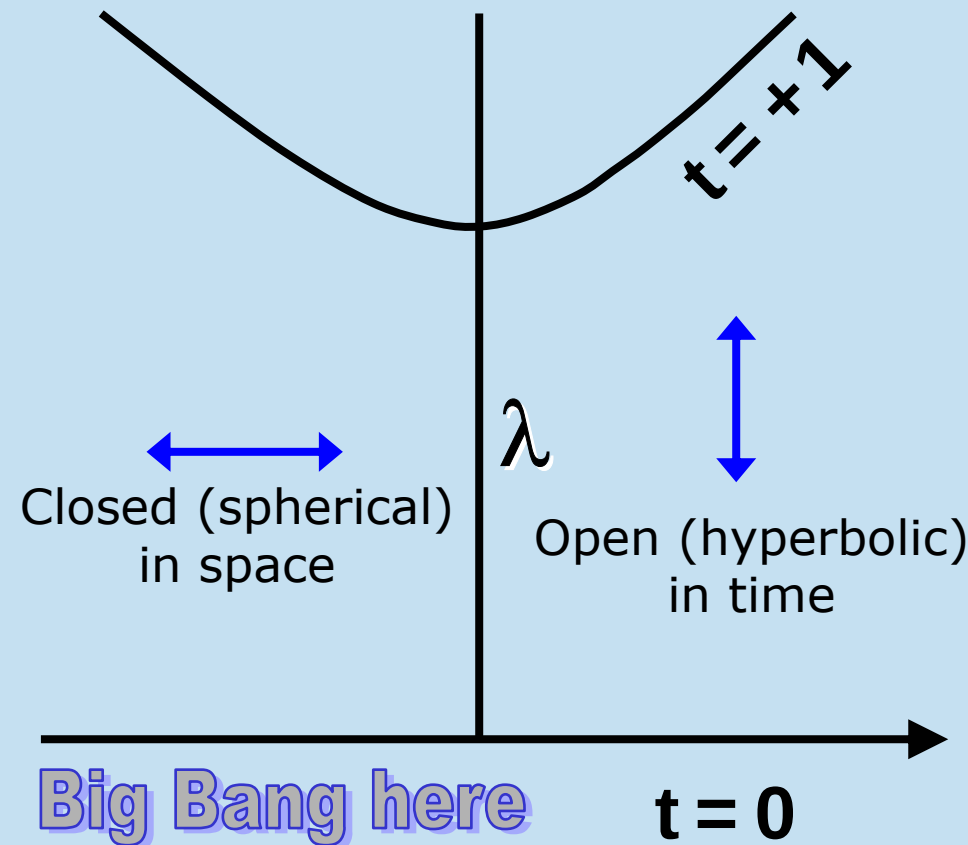
What is Λ ?

- More directly, de Sitter space has boundaries as shown
- Cosmological constant in this space is $\Lambda = 12/\lambda^2$
- Bigger the space is (in space and time), the smaller Λ is
- Also the Hubble constant arises geometrically:
 $H = H_0 = 2/\lambda$
- Could our actual universe (which has a big bang) be fitted into such a diagram?



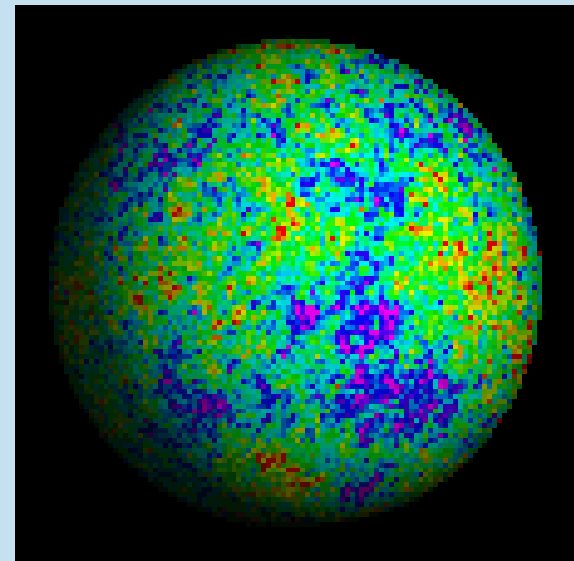
Combining Big Bang and de Sitter

- Want a Big Bang origin, but then tending to CGA version of de Sitter in future
- Amounts to a boundary condition on how far a photon is able to travel by the end of the universe!
- Find can satisfy this, but (big surprise) only works for a particular flow line!
- Says current universe has $\Omega_{\text{total}} \approx 1.10$ i.e. closed (has to be to match spatial curvature of de Sitter)



Does it work?

- Problem: starting with CMB data from end last year (e.g. Cambridge Very Small Array data!) appears unlikely that universe can be more than 5% closed
- Recent Wilkinson Microwave Anisotropy Probe data, and Hubble constant determination from Hubble Space Telescope, confirm this



Origin of the fluctuations

However, this has ignored the question of how the fluctuations (CMB + matter) get there

- Current theory is that they were produced during a period of **inflation** in the very early universe
- Basically “inflation” just means acceleration
- Universe inflates about 10^{22} times in a tiny fraction of a second
- Tiny quantum fluctuations get amplified to the scale of galaxies and clusters



Scalar fields

- To drive this, turns out we need negative pressure
- Only something called a **scalar field** can provide this
 - basically just need a scalar particle with mass
- So have to put a scalar field into our CGA approach!
- Works amazingly well! Gives a quantitative link between the amount of inflation in the early universe, and how small the cosmological constant is today
- Predicts present $\Omega \approx 1.02-1.04$, i.e. Universe is just closed spatially
- Fits in fine with the WMAP and latest large scale structure measurements, and may resolve some problems with these on both large and small scales

Acknowledgements



Joan and Robert Lasenby
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