The Fast Fourier Transform

Chris Lomont, Jan 2010, http://www.lomont.org, updated Aug 2011 to include parameterized FFTs.

This note derives the Fast Fourier Transform (FFT) algorithm and presents a small, free, public domain implementation with decent performance. A secondary goal is to derive and implement a real to complex FFT usable in most cases where engineers use complex to complex FFTs, which gives double the performance of the usual complex to complex FFT while retaining all the output information. A final goal is to show how to use the real to complex FFT as a basis for sound processing tasks such as creating frequency spectrograms.

This work was necessitated by work on the HypnoCube, www.hypnocube.com, which required a small, free, public domain, decently performing FFT implementation in C#. Since all the FFT implementations easily found failed on at least one aspect of this, this implementation was created and is released at the end of this document. The rest of this note details the real to complex FFT construction and how to apply this faster, lower resource using FFT for sound processing.

Introduction

The Fourier Transform converts signals from a time domain to a frequency domain and is the basis for many sound analysis and visualization algorithms. It converts a signal into magnitudes and phases of the various sine and cosine frequencies making up the signal.

For example, take the signal shown in Figure 1 from the formula also shown in Figure 1, and sample it at the 32 equally spaced points shown.

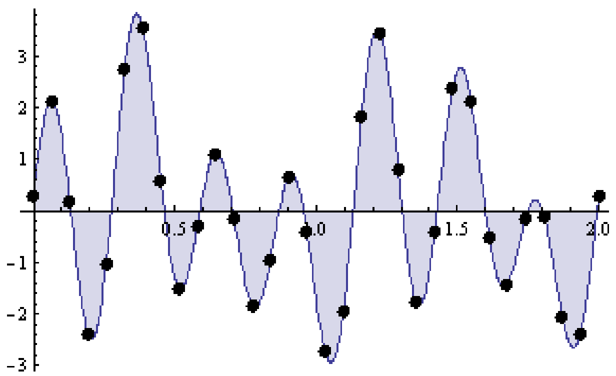


Figure 1: $\frac{1}{\sqrt{2}}\sin 2\pi t - \frac{1}{\sqrt{2}}\cos 2\pi t + \cos 5\pi t + 2\sin 7\pi t$

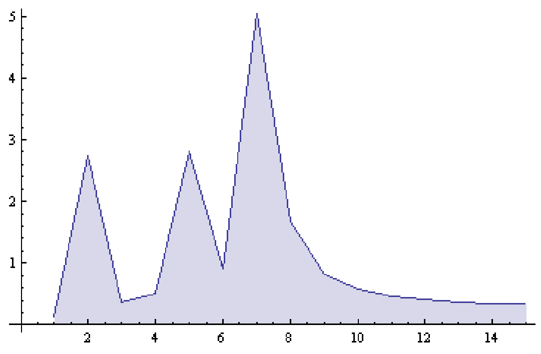


Figure 2 - The Fourier Transform of the samples

Taking the Fourier Transform of these 32 real valued points results in 32 complex valued points. As shown later, these 32 points have a symmetry, so plotting only the magnitudes of the first 16 gives the result in Figure 2; the other 16 points are the mirror image due to the symmetry.

Notice spikes at entries 2, 5, and 7, which correlate to the periods of the components of the input function. The complex value associated to the magnitude at 2 is $-1.3787 + 2.35648\,i$. The real and imaginary parts are of similar magnitude and are larger than surrounding values. The similarity in magnitude is because the phase of the $2\pi t$ component comes from two equal amplitude parts. The component of the period 5π piece is $2.61789 - 1.00959\,i$, where the real and imaginary parts have slightly more separation in magnitude. This is because the phase of that part is "less" out of sync with the overall period of the signal. This combination of magnitude and phase in each output entry gives the "strength" and relationship between the various frequencies underlying the signal.

The Fast Fourier Transform (FFT)

Different applications define the Fourier Transform slightly differently, mostly relating to signs and scaling factors. To accommodate many different uses, the formula given next represents many different conventions. Given N (real or complex) samples $x_0, x_1, x_2, \dots, x_{N-1}$, the Fourier Transform of them are the N complex numbers $y_0, y_1, y_2, \dots, y_{N-1}$ given by[1]

$$y_j = \frac{1}{N^{(1-a)/2}} \sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} bjk} x_k \quad , \quad \text{for } j = 0,1,\dots,N-1$$

With this definition with integer parameters a and b the inverse Fourier Transform is given by

$$x_k = \frac{1}{N^{(1+a)/2}} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} bjk} y_j \quad , \quad \text{for } k = 0,1,\dots,N-1$$

The parameters a and b are integers allowing changing the initial scaling term and the sign on the exponential. $|b|$ should be relatively prime to N. Common values for (a, b) are $(0,1)$ used by mathematicians, $(-1,1)$ for data analysis, and $(1,-1)$ for signal processing. The previous version of this note was derived using $(1,1)$ which led to user confusion.

These complex numbers represent the magnitude and phase of the various frequencies present in the x_k. A direct implementation of this requires adding N values for each of N array entries, for a complexity of $O(N^2)$. Later this is tested as the "naive" implementation. If this was the best that could be done then the Fourier Transform would not have nearly as much impact on algorithms and computer science as it does.

Given N input points, the Fast Fourier Transform (FFT) computes the Fourier Transform in $O(N \log N)$ steps. The majority of this note derives the FFT algorithm and shows how to implement if efficiently.

Let $\omega_N = e^{\frac{2\pi i}{N} b}$ to simplify notation. Note we put the b parameter into here. Let the scaling terms be written as $s_N^+ = \frac{1}{N^{(1-a)/2}}$ for the forward transform and $s_N^- = \frac{1}{N^{(1+a)/2}}$ for the inverse transform. Then the Fourier Transform can be rewritten (for even N) by splitting into even and odd x indices as

[1] Some places define it differently mostly relating to signs and factors of N. All the ideas explained in this note can be adapted to other definitions.

$$\begin{aligned}
y_j &= s_N^+ \sum_{k=0}^{N-1} \omega_N^{jk} x_k \\
&= s_N^+ \sum_{k=0}^{N/2-1} \omega_N^{j2k} x_{2k} + s_N^+ \sum_{k=0}^{N/2-1} \omega_N^{j(2k+1)} x_{2k+1} \\
&= s_N^+ \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k} + s_N^+ \omega_N^j \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k+1} \\
&= 2^{-(1-a)/2} (F_{j,N/2}^e + \omega_N^j F_{j,N/2}^o)
\end{aligned}$$

where $F_{j,N/2}^e$ is the transform of the $N/2$ even indexed data points and $F_{j,N/2}^o$ is the transform of the odd ones[2]. The value of $2^{-(1-a)/2}$ in front comes from considering the scaling factors s_N^+ and $s_{N/2}^+$. If we take N as a power of 2, say $N = 2^n$, this reducting from N to $N/2$ can be repeated until all terms look like $F_{j,1}^{eoe\ldots eeoo}$ and each is represented by one term, say $F_{j,1}^{eoe\ldots eeoo} = x_q$ for some q, since the Fourier Transform of a single point is itself.

To see which q corresponds with which $eoe \ldots eeoo$, expand the $N = 8$ case and watch what happens. To make it clearer we skip writing the scaling factors, but we have to remember them when we implement. Evaluate $F_{5,8}$, that is, the 5th term

$$\begin{aligned}
F_{5,8} &= F_{5,4}^e + \omega_8^5 F_{5,4}^o \\
&= \left(F_{5,2}^{ee} + \omega_4^5 F_{5,2}^{eo}\right) + \omega_8^5 \left(F_{5,2}^{oe} + \omega_4^5 F_{5,2}^{oo}\right) \\
&= \left(\left(F_{5,1}^{eee} + \omega_2^5 F_{5,1}^{eeo}\right) + \omega_4^5 \left(F_{5,1}^{eoe} + \omega_2^5 F_{5,1}^{eoo}\right)\right) + \omega_8^5 \left(\left(F_{5,1}^{oee} + \omega_2^5 F_{5,1}^{oeo}\right) + \omega_4^5 \left(F_{5,1}^{ooe} + \omega_2^5 F_{5,1}^{ooo}\right)\right) \\
&= \left(\left(x_{eee} + \omega_2^5 x_{eeo}\right) + \omega_4^5 \left(x_{eoe} + \omega_2^5 x_{eoo}\right)\right) + \omega_8^5 \left(\left(x_{oee} + \omega_2^5 x_{oeo}\right) + \omega_4^5 \left(x_{ooe} + \omega_2^5 x_{ooo}\right)\right) \\
&= \left(\left(x_0 + \omega_2^5 x_4\right) + \omega_4^5 \left(x_2 + \omega_2^5 x_6\right)\right) + \omega_8^5 \left(\left(x_1 + \omega_2^5 x_5\right) + \omega_4^5 \left(x_3 + \omega_2^5 x_7\right)\right)
\end{aligned}$$

In the last two lines the x_j are placed back in using both the e and o notation and also with normal subscripts. Notice these are related by the following patterns.

Input Order		*Binary*		*Reversed*		*Output Order*
0	→	000	→	000	→	0
1	→	001	→	100	→	4
2	→	010	→	010	→	2
3	→	011	→	110	→	6
4	→	100	→	001	→	1
5	→	101	→	101	→	5
6	→	110	→	011	→	3
7	→	111	→	111	→	7

The left column is the original index ordering and the right hand column is the new ordering. The in between columns are the values in binary, which match the e and o patterns. *Notice the middle arrows are the reversal of the binary values on their left and right.*

[2] Note that these are valid for any j values by making the x_k repeat and using the periodicity of powers of ω_N.

It appears that the q value is obtained by taking the sequence of $eoe \dots eeoo$, replacing $e = 0$ and $o = 1$, reversing the digits, and using this as the binary value for q. Careful analysis shows this pattern holds up in general.

So, from the last equation, the Fourier Transform can be obtained for $N = 8$ by doing a 2-point Fourier Transform on x_0 and x_4 , x_2 and x_6 , x_1 and x_5 , and x_3 and x_7. Then a 4-point can be done on the results, on indices 0,4,2,6 and 1,5,3,7. Finally these can be merged over all 8 indices in the order 0,4,2,6,1,5,3,7.

Implementing this carefully gives the basic 8 point FFT algorithm consisting of the following steps:

1. Sort the inputs into bit reversed index order, which evaluates the eight $F_{j,1}$ transforms.
2. Evaluate four $F_{j,2}$ transforms on pairs from step 1.
3. Evaluate two $F_{j,4}$ transforms on data from step 2.
4. Evaluate one $F_{j,8}$ transform on data from step 3.

The general size $N = 2^n$ transform follows the same pattern, using 1 step of bit reversal (in time $O(N)$ as shown below), and then $n = \log N$ steps, each using $O(N)$ steps to compute for $j = 0,1,2,\dots,N-1$ outcomes. Thus the result is an $O(N \log N)$ time algorithm. To do this from the last set of equations takes careful ordering of how the transforms get done. This algorithm is called the Fast Fourier Transform, or FFT.

The first step, bit reversing the indices, has an elegant solution. Implementing a bit counter (0,0,0),(0,0,1),(0,1,0),..., using binary, a carry, and an array to track digits, is straightforward. The trick is to reverse the array, counting from the left, and create a counter that counts with the digits reversed. This is then improved to work without arrays by keeping two counters j and k counting from opposite directions. Knuth's The Art Of Computer Programming (TAOCP), has a solution in exercise 5, section 7.2.1.1, as follows:

Input:	An array of 2^n values.
Output:	The array with item i swapped with item j if the binary representation of i is the reverse of the binary representation of j.
Step R1: Initialize	Set $j = k = 0$.
Step R2: Swap	Swap entries $j + 1$ and $k + 2^{n-1}$. If $j > k$ swap items j and k and also swap items $j + 1 + 2^{n-1}$ and $k + 1 + 2^{n-1}$.
Step R3: Advance k	Set $k = k + 2$. If $k \geq 2^{n-1}$ then stop.
Step R4: Advance j	Let $h = 2^{n-2}$. while $j \geq h$ set $j = j - h$ and $h = h/2$. Finally set $j = j + h$. Goto step R2.

One final technique is used. In computing the FFT many powers of ω_N are used. Instead of computing them each time one is needed, which requires expensive sine and cosine computations, they can be computed using a simple recurrence.

Computing the various ω_N^j can be done by successively multiplying ω_N. Using Euler's expansion $\omega_N = e^{\frac{2\pi i}{N}b} = \cos\frac{2\pi b}{N} + i\sin\frac{2\pi b}{N}$ this can be computed using complex multiplication with a recurrence relation. Using real variables this can be accomplished in $O(N)$ steps via:

Input:	$N = 2^n$, an integral power of 2.
Output:	The powers of $\omega_N = e^{\frac{2\pi i b}{N}} = \cos\frac{2\pi b}{N} + i\sin\frac{2\pi b}{N}$ at various points of the algorithm.
P1: Initialize	Set $x = 1, y = 0$ (used as the initial 0th power of ω_N), $\alpha = \cos\frac{2\pi b}{N}$ and $\beta = \sin\frac{2\pi b}{N}$ (the increment ω_N), and $j = 0$ (the current power).
P2: Increment	Set $t \leftarrow x, x \leftarrow \alpha x - \beta y, y \leftarrow \beta t + \alpha y, j \leftarrow j + 1$. Now $x + iy$ equals ω_N^j.
P3: Loop	If $j < N - 1$ goto step P2, else end.

There is a slight variant used for increasing numerical robustness. Real numbers are necessarily represented with finite precision on computers, and for large values of N the value of $\cos\frac{2\pi}{N}$ is very close to 1. With limited number of digits stored using a binary form of scientific notation, this loses precision. For example, as the value gets close to 1, say 0.9999, then 0.99999, ... it eventually will round to 1. It is more precise to store the value $1 - \cos\frac{2\pi}{N}$, since in scientific notation more digits will be retained. This is computed with the trig identity $1 - \cos\frac{2\pi}{N} = 2\sin^2\frac{\pi}{N}$, avoiding computations that lose significant digits. Computing $\sin\frac{2\pi}{N}$ does not suffer this problem since it is already small.

This leads to a slightly more costly to perform algorithm, but one that is numerically better behaved for large N. It uses two more additions per loop than the simpler algorithm, but is still $O(N)$.

Input:	$N = 2^n$, an integral power of 2.
Output:	The powers of $\omega_N = e^{\frac{2\pi i b}{N}} = \cos\frac{2\pi b}{N} + i\sin\frac{2\pi b}{N}$ at various points of the algorithm.
Q1: Initialize	Set $x = 1, y = 0$ (used as the initial 0th power of ω_N) $\alpha = 2\sin^2\frac{\pi b}{N}$ and $\beta = \sin\frac{2\pi b}{N}$, and $j = 0$ (the current power).
Q2: Increment	Set $t \leftarrow x, x \leftarrow \alpha x - \beta y + x, y \leftarrow \beta t + \alpha y + y, j \leftarrow j + 1$. Now $x + iy$ equals ω_N^j.
Q3: Loop	If $j < N - 1$ goto step P2, else end.

Putting all this together, one obtains a decent FFT algorithm. Careful ordering of indices and the work gives the following algorithm which transforms the data in place. Assume variables can hold complex numbers for now, which will be changed later to standard use of real valued variables.

The following algorithm implements this in pseudocode. It works on complex numbers and does the computation in place, that is, only using a constant amount of extra storage for any value of N. The letter i is the complex unit in the code.

Input:	DATA, an array of $N = 2^n$ complex numbers $x_0, x_1, x_2, \ldots, x_{N-1}$.
Output:	N complex values $y_0, y_1, y_2, \ldots, y_{N-1}$ which are the Fourier Transform of the input.

```
BitReverseData(data)
mmax = 1
while (n > mmax)
   istep = 2 * mmax
   theta = 3.14159265358979323846 * b / mmax
   wp = cos(theta) + i * sin(theta)
   w = 1
   for (m = 1; m <= mmax; m = m + 1)
```

```
        for (k = m - 1; k < n; k = k + istep)
           j = k + mmax
           temp = w * x[j]
           x[j] = x[k] - temp
           x[k] = x[k] + temp
        endfor
     w = w*wp
     endfor
     mmax = istep
  endwhile
  ScaleData(data)
```

This same pseudo code can be used to do the inverse transform by negating theta and changing the scaling appropriately, allowing merging of FFT and inverse FFT routines into one routine.

Performance Improvements

The above algorithm is tested on a modern PC (circa 2010) in several forms and against a few other FFT implementations. All source code is available for testing on the author's website, www.lomont.org. The implementations (all tested in C# since that was the driving need for this work) are

1. The "naive" algorithm with running time $O(N^2)$ listed above. This is clearly very slow.

2. The algorithm presented in Numerical Recipes[3], a common book of numerical algorithms. Their license restrictions make this code unusable in almost any application.

3. The NAudio[4] library on CodePlex, a C# audio processing library. The FFT routines rely on other parts of the library, and is much slower then the final real to complex FFT listed below.

4. Lomont1FFT, an implementation of the above algorithm using a lightweight complex number class. Real valued data is read into the FFT function in an array of alternating real and complex values, converted to complex numbers, passed through the FFT, and converted back. This is quite inefficient.

5. Lomont2FFT, which unrolls the complex numbers by replacing each complex entry with two real entries, but using indices from the above code, resulting in many multiplies by two to access real components and a multiply by 2 and add one for imaginary components. This is slightly faster.

6. Lomont3FFT, which re-derives the algorithm indices to avoid all the multiplies by two. This and the next version were the best performing.

7. Lomont4FFT, which pre-computes the sine and cosine values needed and uses them across multiple runs. This is useful when one does many FFTs of the same size in an application, which

[3] http://www.nr.com/
[4] http://www.codeplex.com/naudio

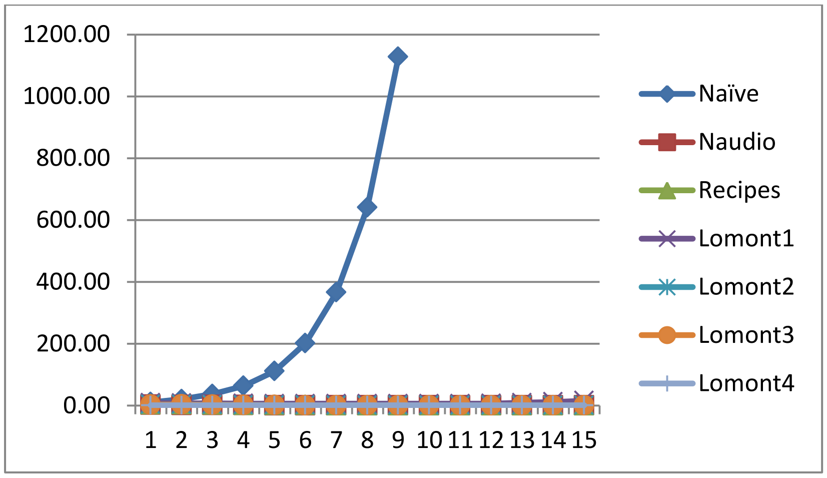
is quite common. This is often the fastest method for doing many applications of the same size FFT.

This testing does not include FFTW (The "Fastest Fourier Transform in the West"), which is a very high performance FFT implementation for any size transform. The goal of this work was to create a simple, single file, small FFT implementation suitable for many projects. FFTW is over 40,000 lines of code. The result of this work is well under 400 lines of code, including an FFT, an inverse FFT, a table based version of each, a real to complex FFT (covered below), and its inverse, all in a single standalone source file.

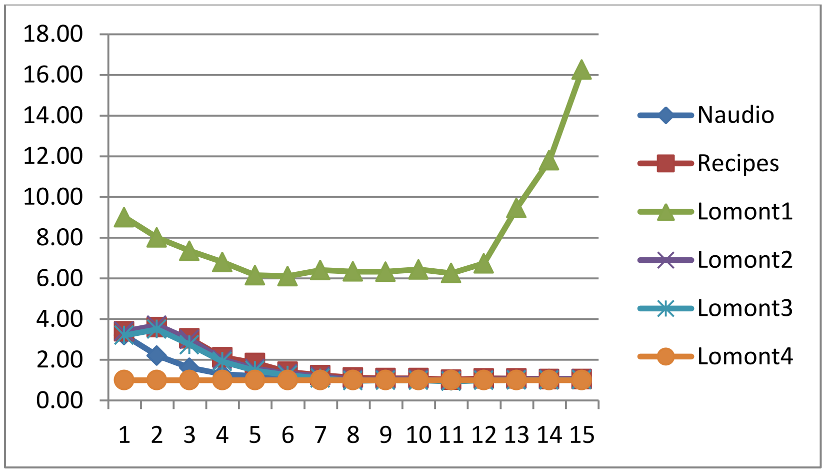
The routines were tested on various sizes of N. For each size, the algorithm is run 100 times to load all items into cache and otherwise smooth out results. Then a loop of 100 applications of the FFT is timed for 50 passes. The min, max, and average of these 50 scores are kept for each N and FFT type combination. Sizes for N were chosen to be small, around those needed in audio processing.

All the following graphs show relative time to Lomont4FFT performance, which was most often the best performing, with the x-axis being the powers of 2 for the size of the FFT. The values used for the ratios are the minimum time taken across the runs, which in each case was very near the average run time. The max run time had noise due to OS interruptions such as thread scheduling, etc. The min time is the best time the algorithm could do the FFT, which is a pretty good measure of its performance.

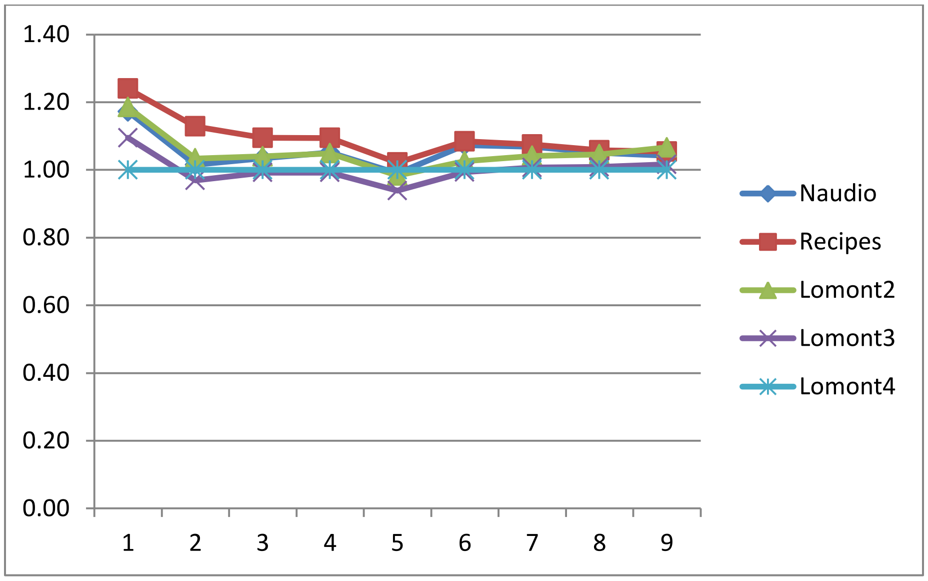
The first graph shows the naive implementation is (as expected) so bad that it drowns out all the other graph information, so it is removed from all further graphs.



The next graph shows Lomont1FFT is much slower than the other implementations, and that for small powers of 2 the Lomont4FFT is much faster than all the others (since the tabled sine and cosine values make a big difference for small N). However at $N = 2^7$ and higher performance is close, so...



The final graph only has sizes 2^7 to 2^{15} (labeled as 1 through 9) listed to see closer performance. This shows how close they all are to baseline Lomont4FFT. Note however that NAudio and Numerical Recipes versions are always slower, but Lomont3 is often faster by a few percent.



Based on this Lomont3FFT and Lomont4FFT are merged into the final FFT library listed at the end of this note.

Real Valued FFT

In many applications, such as sound processing, the x_k are real valued, which allows a performance and space improvement over the complex input to complex output FFT. When implementing the FFT in a many computer languages, complex numbers are stored as two floating point values, one for the real component and one for the imaginary component. To perform an FFT which expects complex inputs, the real valued input has to be augmented by an equal amount of space for the imaginary components, which takes additional time and space.

For real input the output of an FFT is still complex valued, so this FFT is called a real to complex FFT. The real to complex FFT performs about twice as fast as the complex to complex FFT on the same input data.

For this section, assume the x_k are real valued. Taking complex conjugates of the FFT gives

$$y_j^* = s_N^+ \sum_{k=0}^{N-1} \omega_N^{-jk} x_k^* = s_N^+ \sum_{k=0}^{N-1} \omega_N^{(N-j)k} x_k = y_{N-j}$$

for $j = 1,2,\ldots,N-1$, thus the output has (conjugate) symmetry; only half of the output is needed to describe the complex output (for real valued input). We used here that $\omega_N^N = 1$ and x_k^*=x_k. y_0 is not on the symmetric list, but since $y_0 = s_N^+ \sum_{k=0}^{N-1} x_k$ it is real valued. $y_{N/2}$ is its own symmetric partner, thus it has to be real (alternatively check $y_{N/2} = s_N^+ \sum_{k=0}^{N/2-1} x_{2k} - s_N^+ \sum_{k=0}^{N/2-1} x_{2k+1}$ which is real).

In short, for real input, the entire complex valued output can be described with only half the output as in the complex case.

The main idea of the real to complex FFT is this: instead of using an array with alternating real and imaginary components as input (wasting half the space since the imaginary half is 0 on input), leave the array filled with the real components only, do the complex to complex FFT treating the array as an array of $N/2$ complex values, and then cleverly reconstruct the desired output. The net result turns out to be a performance and space gain.

Now for details.

Take the real sequence $x_0, x_1, x_2, \ldots, x_{N-1}$ of length $N = 2^n$ and treat it as a sequence of $N/2$ complex values $z_k = x_{2k} + ix_{2k+1}$ by treating the x_k as interleaved complex values. Then perform a FFT on the z_k of half the length. This gives

$$\begin{aligned} t_j &= s_{N/2}^+ \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} (x_{2k} + ix_{2k+1}) \text{ for } j = 0,1,\ldots,N/2-1 \\ &= s_{N/2}^+ \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k} + is_{N/2}^+ \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k+1} \end{aligned}$$

Compare this to an earlier expression for the desired output:

$$y_j = s_N^+ \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k} + s_{N/2}^+ \omega_N^j \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k+1}$$

The similarity leads to an efficient algebraic solution for the y_j in terms of the t_j. First, using the fact the x_k are real, note that (after some algebra)

$$t_{N/2-j}^{*} = s_{N/2}^{+} \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k} - i s_{N/2}^{+} \sum_{k=0}^{N/2-1} \omega_{N/2}^{jk} x_{2k+1} \text{ for } j = 1,2,\dots,N/2-1$$

Solving for the y_j gives, after a little simple algebra (or check by direct substitution)

$$\begin{aligned} y_j &= \frac{\delta}{2}\left(t_j + t_{N/2-j}^{*}\right) - \frac{i\delta}{2}\left(t_j - t_{N/2-j}^{*}\right)\omega_N^j \quad \text{for} \quad j = 1,2,\dots,N/2-1 \\ y_0 &= \frac{\delta}{2}(t_0 + t_0^{*}) - \frac{i\delta}{2}(t_0 - t_0^{*}) \\ y_{N/2} &= \frac{\delta}{2}(t_0 + t_0^{*}) + \frac{i\delta}{2}(t_0 - t_0^{*}) \end{aligned}$$

Where $\delta = s_N^{+}/s_{N/2}^{+}$ is a correction to align the scaling factors. This reconstructs the desired y_j given the transform t_j of the packed z_k. The final theoretical question is how to do the inverse transform: this amounts to solving for the t_j in terms of the y_j and then doing the inverse FFT on the t_j, obtaining the original packed x_j. Simple algebra reconstructs the t_j :

$$\begin{aligned} t_j &= \frac{1}{2\delta}\left(y_j + y_{N/2-j}^{*}\right) + \frac{i}{2\delta}\left(y_j - y_{N/2-j}^{*}\right)\omega_N^{-j} \quad \text{for} \quad j = 1,2,\dots,N/2-1 \\ t_0 &= \frac{1}{2\delta}\left(y_0 + y_{N/2}\right) + \frac{i}{2\delta}\left(y_0 - y_{N/2}\right) \end{aligned}$$

All the steps are very similar to the forward transform, and can be accomplished with the same code with minor sign changes.

Since each y_j is computed from two t_j that are paired in a symmetric manner, both y_j and $y_{N/2-j}$ can be computed at together in place, thereby requiring only constant space overhead. Doing it in place makes the entire operation faster.

For output, each $y_j, j = 1,2,\dots,N/2-1$ uses the two spots where the x_{2j}, x_{2j+1} initially were. The real valued y_0 and $y_{N/2}$ will be packed into the slots where x_0 and x_1 were.

The powers of ω_N^j are again computed using the recurrence idea from the FFT. All this is implemented as RealFFT in the final code, which performs about twice as the fastest complex to complex FFT across the sizes tested above for real input.

Sound Analysis

To use the FFT for sound analysis, it is necessary to understand the output of the FFT. This section carefully explains the input and output of the RealFFT.

For input, the samples are assumed to be equally spaced in time. Common sample bit-depths are 16-bit, 20-bit, or 24-bit audio. For example, for 16-bit signed audio samples, values are integers in the range $[-32768,\ 32767]$. The sampling rate S is the number of samples per second, with common rates

44100 Hz and 22050 Hz. Define $\Delta = \frac{1}{S}$ to be the time elapsed between samples. Then $N = 2^n$ samples are taken at times $t = 0,\ \Delta,\ 2\Delta,\ 3\Delta, \dots,\ (N-1)\Delta$, as shown in Table 1 below. Samples are real numbers (actually in this case integers) and are going to be passed to the RealFFT routine for processing.

The output from the RealFFT consists of the first two entries which have some special properties, and then $N-2$ values in pairs which are the real and imaginary parts of the FFT values. These encode the phase and magnitude[5] of each frequency from $f = \frac{1}{N\Delta}, \frac{2}{N\Delta}, \frac{3}{N\Delta}, \frac{4}{N\Delta}, \frac{(N/2-1)}{N\Delta}$. These are returned in indices 2 and higher. The output range can be up to N times the input range, which can be scaled after the transform to keep values in desired ranges. For example, a $N = 512$ sample real FFT on 16-bit signed values can have output range in $[-32768 * 512,\ 32767 * 512]$ = [-16731136, 16730624].

The lowest index of the output data, recall, is the sum of all the inputs, and the second lowest indexed output is the sum of all the even index inputs minus the sum of all the odd index inputs. These lowest two values are often ignored for frequency spectrograms.

This means the lowest frequency recognized is $f = \frac{S}{N}$ and the highest one is $f = \frac{S}{N}(N/2-1) \approx \frac{S}{2}$. Since N samples are required for each RealFFT, there is a latency of $\frac{N}{S}$ seconds from the input sound to analysis results.

Input/Output Array index.	**Input meaning, each a real number.**	**Output frequencies, alternating real and imaginary parts after the first two entries.**
0	$t = 0$	$f = 0$ term, sum of all inputs, real.
1	$t = \Delta$	$f = \pm\frac{1}{2\Delta}$ term, real
2	$t = 2\Delta$	$f = \frac{1}{N\Delta}$, real part
3	$t = 3\Delta$	$f = \frac{1}{N\Delta}$, imaginary part
4	$t = 4\Delta$	$f = \frac{2}{N\Delta}$, real part
5	$t = 5\Delta$	$f = \frac{2}{N\Delta}$, imaginary part
…	…	…
$\boldsymbol{N-2}$	$t = (N-2)\Delta$	$f = \frac{N/2-1}{N\Delta}$, real part
$\boldsymbol{N-1}$	$t = (N-1)\Delta$	$f = \frac{N/2-1}{N\Delta}$, imaginary part

Table 1 - Input and Output meaning for the RealFFT.

For example, to analyze frequencies from a piano (27.5 Hz to 4186 Hz) a maximal frequency around $\frac{S}{2} = 4186$ is needed, giving $S = 8372$ samples per second. The minimal frequency of 27.5 then gives $27.5 = \frac{S}{N}$ giving $N = 305$. N must be a power of 2, so taking the next highest one $N = 512$ gives a lowest frequency response of 16 Hz.

To analyze music more accurately to higher frequencies, raise the sample rate. Taking the common $S = 22050$ Hz with a $N = 1024$ sample size, 1023 frequencies from 21.5 Hz to 11025 Hz are returned, with a latency of $1024/22050 = 0.0464$ seconds.

[5] Recall the magnitude of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$ (for those not well versed in complex analysis).

The magnitudes of each frequency band give the values needed for a spectrum analyzer. To make larger bands just aggregate these into the desired number of bands in the desired ranges. This is how to do basic sound analysis, and code is given next for the various FFTs, one of the most complicated pieces.

The Code

In the code, function FFT implement a standard complex to complex FFT, implementing Lomont3FFT from testing above. This is the routine to use for a single use complex to complex FFT. The table based Lomont4FFT algorithm tested above is implemented in TableFFT; it automatically creates the table on first call, making subsequent calls faster. This is the complex to complex to use for repeated FFTs of the same size. RealFFT is the real to complex FFT most useful for many needs, such as sound processing. It uses the TableFFT internally; holding the object between calls is necessary to increase performance. Each transform takes a data array of doubles and a boolean denoting forward or inverse transforms.

Finally, the FFT routine uses the faster, less accurate recurrence, which is fine for values of $N \leq 10^9$ (since an IEEE 754 double has a 53 bit mantissa - another story). This could be replaced with the recurrence from the Initialize routine to make the FFT more accurate but slightly slower.

The website http://www.lomont.org has a source file for this along with a unit test.

```
// Code to implement decently performing FFT for complex and real valued
// signals. See www.lomont.org for a derivation of the relevant algorithms
// from first principles. Copyright Chris Lomont 2010-2012.
// This code and any ports are free for all to use for any reason as long
// as this header is left in place.
// Version 1.1, Sept 2011
using System;

/* History:
 * Sep 2011 - v1.1 - added parameters to support various sign conventions
 *                   set via properties A and B.
 *                 - Removed dependencies on LINQ and generic collections.
 *                 - Added unit tests for the new properties.
 *                 - Switched UnitTest to static.
 * Jan 2010 - v1.0 - Initial release
 */
namespace Lomont
{
    /// <summary>
    /// Represent a class that performs real or complex valued Fast Fourier
    /// Transforms. Instantiate it and use the FFT or TableFFT methods to
    /// compute complex to complex FFTs. Use FFTReal for real to complex
    /// FFTs which are much faster than standard complex to complex FFTs.
    /// Properties A and B allow selecting various FFT sign and scaling
    /// conventions.
    /// </summary>
    public class LomontFFT
    {
        /// <summary>
        /// Compute the forward or inverse Fourier Transform of data, with
        /// data containing complex valued data as alternating real and
        /// imaginary parts. The length must be a power of 2. The data is
        /// modified in place.
        /// </summary>
        /// <param name="data">The complex data stored as alternating real
        /// and imaginary parts</param>
        /// <param name="forward">true for a forward transform, false for
        /// inverse transform</param>
        public void FFT(double[] data, bool forward)
        {
            var n = data.Length;
            // checks n is a power of 2 in 2's complement format
            if ((n & (n - 1)) != 0)
                throw new ArgumentException(
                    "data length " + n + " in FFT is not a power of 2");
            n /= 2;    // n is the number of samples

            Reverse(data, n); // bit index data reversal

            // do transform: so single point transforms, then doubles, etc.
            double sign = forward ? B : -B;
            var mmax = 1;
            while (n > mmax)
            {
                var istep = 2 * mmax;
                var theta = sign * Math.PI / mmax;
                double wr = 1, wi = 0;
                var wpr = Math.Cos(theta);
                var wpi = Math.Sin(theta);
                for (var m = 0; m < istep; m += 2)
                {
```

```
                for (var k = m; k < 2 * n; k += 2 * istep)
                {
                    var j = k + istep;
                    var tempr = wr * data[j] - wi * data[j + 1];
                    var tempi = wi * data[j] + wr * data[j + 1];
                    data[j] = data[k] - tempr;
                    data[j + 1] = data[k + 1] - tempi;
                    data[k] = data[k] + tempr;
                    data[k + 1] = data[k + 1] + tempi;
                }
                var t = wr; // trig recurrence
                wr = wr * wpr - wi * wpi;
                wi = wi * wpr + t * wpi;
            }
            mmax = istep;
        }

        // perform data scaling as needed
        Scale(data,n, forward);
    }

    /// <summary>
    /// Compute the forward or inverse Fourier Transform of data, with data
    /// containing complex valued data as alternating real and imaginary
    /// parts. The length must be a power of 2. This method caches values
    /// and should be slightly faster on than the FFT method for repeated uses.
    /// It is also slightly more accurate. Data is transformed in place.
    /// </summary>
    /// <param name="data">The complex data stored as alternating real
    /// and imaginary parts</param>
    /// <param name="forward">true for a forward transform, false for
    /// inverse transform</param>
    public void TableFFT(double[] data, bool forward)
    {
        var n = data.Length;
        // checks n is a power of 2 in 2's complement format
        if ((n & (n - 1)) != 0)
            throw new ArgumentException(
                "data length " + n + " in FFT is not a power of 2"
                );
        n /= 2;    // n is the number of samples

        Reverse(data, n); // bit index data reversal

        // make table if needed
        if ((cosTable == null) || (cosTable.Length != n))
            Initialize(n);

        // do transform: so single point transforms, then doubles, etc.
        double sign = forward ? B : -B;
        var mmax = 1;
        var tptr = 0;
        while (n > mmax)
        {
            var istep = 2 * mmax;
            for (var m = 0; m < istep; m += 2)
            {
                var wr = cosTable[tptr];
                var wi = sign * sinTable[tptr++];
                for (var k = m; k < 2 * n; k += 2 * istep)
                {
```

```
                var j = k + istep;
                var tempr = wr * data[j] - wi * data[j + 1];
                var tempi = wi * data[j] + wr * data[j + 1];
                data[j] = data[k] - tempr;
                data[j + 1] = data[k + 1] - tempi;
                data[k] = data[k] + tempr;
                data[k + 1] = data[k + 1] + tempi;
            }
        }
        mmax = istep;
    }


    // perform data scaling as needed
    Scale(data, n, forward);
}

/// <summary>
/// Compute the forward or inverse Fourier Transform of data, with
/// data containing real valued data only. The output is complex
/// valued after the first two entries, stored in alternating real
/// and imaginary parts. The first two returned entries are the real
/// parts of the first and last value from the conjugate symmetric
/// output, which are necessarily real. The length must be a power
/// of 2.
/// </summary>
/// <param name="data">The complex data stored as alternating real
/// and imaginary parts</param>
/// <param name="forward">true for a forward transform, false for
/// inverse transform</param>
public void RealFFT(double[] data, bool forward)
{

    var n = data.Length; // # of real inputs, 1/2 the complex length
    // checks n is a power of 2 in 2's complement format
    if ((n & (n - 1)) != 0)
        throw new ArgumentException(
            "data length " + n + " in FFT is not a power of 2"
            );

    var sign = -1.0; // assume inverse FFT, this controls how algebra below works
    if (forward)
    { // do packed FFT. This can be changed to FFT to save memory
        TableFFT(data, true);
        sign = 1.0;
        // scaling - divide by scaling for N/2, then mult by scaling for N
        if (A != 1)
        {
            var scale = Math.Pow(2.0, (A - 1) / 2.0);
            for (var i = 0; i < data.Length; ++i)
                data[i] *= scale;
        }
    }

    var theta = B * sign * 2 * Math.PI / n;
    var wpr = Math.Cos(theta);
    var wpi = Math.Sin(theta);
    var wjr = wpr;
    var wji = wpi;

    for (var j = 1; j <= n/4; ++j)
```

```
        {
            var k = n / 2 - j;
            var tkr = data[2 * k];    // real and imaginary parts of t_k  = t_(n/2 - j)
            var tki = data[2 * k + 1];
            var tjr = data[2 * j];    // real and imaginary parts of t_j
            var tji = data[2 * j + 1];

            var a = (tjr - tkr) * wji;
            var b = (tji + tki) * wjr;
            var c = (tjr - tkr) * wjr;
            var d = (tji + tki) * wji;
            var e = (tjr + tkr);
            var f = (tji - tki);

            // compute entry y[j]
            data[2 * j] = 0.5 * (e + sign * (a + b));
            data[2 * j + 1] = 0.5 * (f + sign * (d - c));

            // compute entry y[k]
            data[2 * k] = 0.5 * (e - sign * (b + a));
            data[2 * k + 1] = 0.5 * (sign * (d - c) - f);

            var temp = wjr;
            // todo - allow more accurate version here? make option?
            wjr = wjr * wpr - wji * wpi;
            wji = temp * wpi + wji * wpr;
        }

        if (forward)
        {
            // compute final y0 and y_{N/2}, store in data[0], data[1]
            var temp = data[0];
            data[0] += data[1];
            data[1] = temp - data[1];
        }
        else
        {
            var temp = data[0]; // unpack the y0 and y_{N/2}, then invert FFT
            data[0] = 0.5 * (temp + data[1]);
            data[1] = 0.5 * (temp - data[1]);
            // do packed inverse (table based) FFT. This can be changed to
            // regular inverse FFT to save memory
            TableFFT(data, false);
            // scaling - divide by scaling for N, then mult by scaling for N/2
            var scale = Math.Pow(2.0, -(A + 1) / 2.0)*2;
            for (var i = 0; i < data.Length; ++i)
                data[i] *= scale;
        }
    }

    /// <summary>
    /// Determine how scaling works on the forward and inverse transforms.
    /// For size N=2^n transforms, the forward transform gets divided by
    /// N^((1-a)/2) and the inverse gets divided by N^((1+a)/2). Common
    /// values for (A,B) are
    ///     ( 0, 1)  - default
    ///     (-1, 1)  - data processing
    ///     ( 1,-1)  - signal processing
    /// Usual values for A are 1, 0, or -1
    /// </summary>
    public int A { get; set; }
```

```
/// <summary>
/// Determine how phase works on the forward and inverse transforms.
/// For size N=2^n transforms, the forward transform uses an
/// exp(B*2*pi/N) term and the inverse uses an exp(-B*2*pi/N) term.
/// Common values for (A,B) are
///     ( 0, 1)  - default
///     (-1, 1)  - data processing
///     ( 1,-1)  - signal processing
/// Abs(B) should be relatively prime to N.
/// Setting B=-1 effectively corresponds to conjugating both input and
/// output data.
/// Usual values for B are 1 or -1.
/// </summary>
public int B { get; set; }

public LomontFFT()
{
    A = 0;
    B = 1;
}

#region Internals

/// <summary>
/// Scale data using n samples for forward and inverse transforms as needed
/// </summary>
/// <param name="data"></param>
/// <param name="n"></param>
/// <param name="forward"></param>
void Scale(double[] data, int n, bool forward)
{
    // forward scaling if needed
    if ((forward) && (A != 1))
    {
        var scale = Math.Pow(n, (A - 1) / 2.0);
        for (var i = 0; i < data.Length; ++i)
            data[i] *= scale;
    }

    // inverse scaling if needed
    if ((!forward) && (A != -1))
    {
        var scale = Math.Pow(n, -(A + 1) / 2.0);
        for (var i = 0; i < data.Length; ++i)
            data[i] *= scale;
    }
}

/// <summary>
/// Call this with the size before using the TableFFT version
/// Fills in tables for speed. Done automatically in TableFFT
/// </summary>
/// <param name="size">The size of the FFT in samples</param>
void Initialize(int size)
{
    // NOTE: if you port to non garbage collected languages
    // like C# or Java be sure to free these correctly
    cosTable = new double[size];
    sinTable = new double[size];
```

```
        // forward pass
        var n = size;
        int mmax = 1, pos = 0;
        while (n > mmax)
        {
            var istep = 2 * mmax;
            var theta = Math.PI / mmax;
            double wr = 1, wi = 0;
            var wpi = Math.Sin(theta);
            // compute in a slightly slower yet more accurate manner
            var wpr = Math.Sin(theta / 2);
            wpr = -2 * wpr * wpr;
            for (var m = 0; m < istep; m += 2)
            {
                cosTable[pos] = wr;
                sinTable[pos++] = wi;
                var t = wr;
                wr = wr * wpr - wi * wpi + wr;
                wi = wi * wpr + t * wpi + wi;
            }
            mmax = istep;
        }
    }

    /// <summary>
    /// Swap data indices whenever index i has binary
    /// digits reversed from index j, where data is
    /// two doubles per index.
    /// </summary>
    /// <param name="data"></param>
    /// <param name="n"></param>
    static void Reverse(double [] data, int n)
    {
        // bit reverse the indices. This is exercise 5 in section
        // 7.2.1.1 of Knuth's TAOCP the idea is a binary counter
        // in k and one with bits reversed in j
        int j = 0, k = 0; // Knuth R1: initialize
        var top = n / 2;  // this is Knuth's 2^(n-1)
        while (true)
        {
            // Knuth R2: swap - swap j+1 and k+2^(n-1), 2 entries each
            var t = data[j + 2];
            data[j + 2] = data[k + n];
            data[k + n] = t;
            t = data[j + 3];
            data[j + 3] = data[k + n + 1];
            data[k + n + 1] = t;
            if (j > k)
            { // swap two more
                // j and k
                t = data[j];
                data[j] = data[k];
                data[k] = t;
                t = data[j + 1];
                data[j + 1] = data[k + 1];
                data[k + 1] = t;
                // j + top + 1 and k+top + 1
                t = data[j + n + 2];
                data[j + n + 2] = data[k + n + 2];
                data[k + n + 2] = t;
                t = data[j + n + 3];
```

```
                    data[j + n + 3] = data[k + n + 3];
                    data[k + n + 3] = t;
                }
                // Knuth R3: advance k
                k += 4;
                if (k >= n)
                    break;
                // Knuth R4: advance j
                var h = top;
                while (j >= h)
                {
                    j -= h;
                    h /= 2;
                }
                j += h;
            } // bit reverse loop
        }

        /// <summary>
        /// Pre-computed sine/cosine tables for speed
        /// </summary>
        double [] cosTable;
        double [] sinTable;

        #endregion

    }
}
// end of file
```